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Abstract

Full Text

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MATHEMATICS

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**ON MIXED BOUNDARY-VALUE PROBLEMS
FOR SINGULAR EQUATIONS OF HYPER-
BOLIC TYPE**

(Presented by Academician S. L. Sobolev on 29 I 1960)

Denote by $u(x, y, \beta, \beta')$ and $\bar{u}(x, y, \beta, \beta')$ the solutions of the equation

$$(y - x)u_{xy} + \beta' u_x - \beta u_y = 0 \quad (0 \leq \beta < 1/2, 0 \leq \beta' < 1/2), \quad (1)$$

belonging to the class L_2 in the domain $D(0 \leq x \leq y \leq x_0)$ of the half-plane $y \geq x$ and satisfying, on two boundaries of this domain, the boundary conditions

$$u(0, y) = 0, \quad u(x, x) = \tau(x), \quad \tau(0) = 0; \quad (2a)$$

$$\bar{u}(0, y) = 0, \quad \bar{u}_\eta(x, x) = \nu(x). \quad (2b)$$

We assume here that $\tau(x)$ and $\nu(x)$ are twice continuously differentiable functions on the interval $(0, x_0)$,

$$\eta = - \left(\frac{y - x}{2 - a - a'} \right)^{1 - \beta - \beta'} \quad (a = 2\beta, a' = 2\beta').$$

Theorem 1. Let the initial values $\tau_1(x)$ and $\tau_2(x)$ be connected by the equality $\tau_2(x) = P(x)\tau_1(x)$, where $P(x)$ is an arbitrary integrable function on $(0, \bar{x}_0)$, $(0 < \bar{x}_0 \leq x_0)$. Then for the corresponding solutions u_1, u_2 of problem (2a), for $\beta'_2 > \beta'_1 \geq 0$,

$$\Gamma(\beta'_2)\Gamma(1 - \beta_1)\Gamma(1 - \beta'_1)\Gamma(1 - \beta_2 - \beta'_2)\chi_1 = -\Gamma(1 - \beta_2)\Gamma(1 - \beta_1 - \beta'_1)$$

the relation

$$u_2(x, y, \beta_2, \beta'_2) = \int_0^x K_1(x, y, \xi, \beta_1, \beta'_1, \beta_2, \beta'_2) u_1(\xi, y, \beta_1, \beta'_1) d\xi, \quad (3)$$

holds, in which

$$K_1 = \chi_1(y-x)^{1-\beta_2-\beta'_2}(y-\xi)^{\beta_1+\beta'_1-1} D_\xi \Omega(\xi),$$

$$\Omega(\xi) = \int_\xi^x P(t)(y-t)^{-\beta}(t-\xi)^{-\beta'_1}(x-t)^{\beta_2-1} dt \quad (D_x = \partial/\partial x, \beta = \beta_1 - \beta_2).$$

An analogous connection formula for $\beta'_2 = \beta'_1$,

$$x_0 \Gamma(1 - \beta_1) \Gamma(1 - \beta_2 - \beta'_1) = \Gamma(1 - \beta_2) \Gamma(1 - \beta_1 - \beta'_1), \quad K_2(\xi) = K_1|_{\beta'_2 = \beta'_1},$$

has the form

$$u_2(x, y, \beta_2, \beta'_1) - x_0 P(x) u_1(x, y, \beta_1, \beta'_1) = \int_0^x K_2(\xi) u_1(\xi, y, \beta_1, \beta'_1) d\xi. \quad (4)$$

If, in particular, $P(x) = 1$, $\beta^* = \beta'_2 - \beta'_1$, $\omega(x - y) = x - \xi$,

$$-\bar{\chi}_1 \Gamma(\beta^*) \Gamma(1 - \beta_1) \Gamma(1 - \beta_2 - \beta'_2) = \Gamma(1 - \beta_2) \Gamma(1 - \beta_1 - \beta'_1),$$

then

$$K_1 = \bar{\chi}_1 (y-x)^{1-\beta_1-\beta'_2} (y-\xi)^{\beta_1+\beta'_1-1} (x-\xi)^{\beta^*-1} F(\beta, \beta'_2, \beta^*; \omega),$$

$$K_2 = -x_0 \beta \beta'_1 (y-x)^{-\beta_1-\beta'_1} (y-\xi)^{\beta_1+\beta'_1-1} F(1 + \beta, 1 + \beta'_1, 2; \omega).$$

The simplest form is assumed by (3) when $\beta_2 = \beta_1$, $\tau_2(x) = \tau_1(x)$. Namely, here

$$\chi_2 \Gamma(\beta') \Gamma(1 - \beta_1 - \beta'_2) = \Gamma(1 - \beta_1 - \beta'_1), \quad \beta'_2 > \beta'_1 \geq 0,$$

$$u_2(x, y, \beta_1, \beta'_2) = \chi_2 (y-x)^{1-\beta_1-\beta'_2} \int_0^x (x-\xi)^{\beta'-1} (y-\xi)^{\beta_1+\beta'_1-1} u_1(\xi, y, \beta_1, \beta'_1) d\xi. \quad (5a)$$

Restricting ourselves to the class of initial values $\tau(x)$ that are $(n + 1)$ -times continuously differentiable on the interval $(0, x_0)$, one may replace (5a) by the expansion

$$u_2(x, y, \beta_1, \beta_2) = x_2 \sum_{k=0}^n \frac{(x-y)^k}{k!} B_{x/y}(k + \beta', 1 - k - \beta_1 - \beta_2) D_x^k u_1(x, y, \beta_1, \beta_1) + R_n, \quad (5b)$$

in which $B_z(p, q) = p^{-1} z^p F(p, 1 - q, 1 + p; z)$ is the incomplete beta function, and

$$R_n = \frac{(-1)^{n+1} x_2}{(n+1)!} (y-x)^{1-\beta_1-\beta_2} \int_0^x \xi^{n+\beta'} (y-x+\xi)^{\beta_1+\beta_1'-1} D_\eta^{n+1} u_1(\eta, y, \beta_1, \beta_1') d\xi$$

$$(\eta = x - \theta\xi, 0 < \theta < 1).$$

In the case of an infinitely differentiable function $\tau(x)$, (5b), as $n \rightarrow \infty$, gives a double series, uniformly and absolutely convergent in the domain D , which in symbolic notation reduces to Humbert's function $\Phi_1(\alpha, \beta, \gamma, X, Y)$ ⁽¹⁾:

$$u_2(x, y, \beta_1, \beta_2) = x_2 \beta' \Phi_1(\beta', 1 - \beta_1 - \beta_1', 1 + \beta'; -\rho, -\delta_x) u_1(x, y, \beta_1, \beta_1'). \quad (6)$$

Here $\rho(y-x) = x$, $\delta_x = x D_x$, $\overline{x_2 \beta'} = x_2$. A direct consequence of the equalities (5) is:

Theorem 2. If on the interval $(0, x_0)$ the derivative $\tau'(x)$ is positive, then, for $\beta = \text{const}$, at any fixed point (x, y) of the domain D the function $u(x, y, \beta, \beta')$ decreases as the parameter β' increases.

Theorem 3. The solutions \bar{u}_1, \bar{u}_2 , for which $\beta_1 > \beta_2 \geq 0$, $\nu_2(x) = P(x)\nu_1(x)$, are related by the equality

$$\bar{u}_2(x, y, \beta_2, \beta_2') = \int_0^x K_3(x, y, \xi, \beta_1, \beta_1', \beta_2, \beta_2') \bar{u}_1(\xi, y, \beta_1, \beta_1') d\xi, \quad (7)$$

where now

$$x_3 \Gamma(\beta_1) \Gamma(\beta_1') \Gamma(1 - \beta_2) \Gamma(\beta_2 + \beta_2') (2 - a_1 - a_1')^{\beta_1 + \beta_1'}$$

$$= -\Gamma(\beta_2') \Gamma(\beta_1 + \beta_1') (2 - a_2 - a_2')^{\beta_2 + \beta_2'},$$

$$K_3 = x_3 D_\xi \int_\xi^x P(t)(x-t)^{-\beta_2}(y-t)^{-\beta'}(t-\xi)^{\beta_1-1} dt.$$

When $\beta_2 = \beta_1$, the left-hand side of formula (7) should be replaced by the expression

$$\bar{u}_2 + x_3 \Gamma(\beta_1) \Gamma(1 - \beta_1) (y - x)^{-\beta'} P(x) \bar{u}_1,$$

and for the values $\nu_2(x) = \nu_1(x)$ one must put there

$$\bar{K}_3 = x_4 (y-x)^{-\beta'} (x-\xi)^{\beta-1} F(\beta', 1-\beta_2, \beta; \omega); \quad x_4 \Gamma(\beta) = -\Gamma(\beta_1) \Gamma(1-\beta_2) x_3.$$

Under transformation of the integrals \bar{u} , as above for u , alongside the case of the unchanged parameter β , one should single out another special case of constant values of β' . Namely, if $\beta'_2 = \beta'_1$, the kernel \bar{K}_3 , being expressed, gives

$$\bar{u}_2(x, y, \beta_2, \beta'_1) = \bar{x}_4 \int_0^x (x-\xi)^{\beta-1} \bar{u}_1(\xi, y, \beta_1, \beta'_1) d\xi \quad (\bar{x}_4 = x_4|_{\beta'=0}). \quad (8a)$$

Assuming here that $\nu(x) \in L_{n+1}(0, x_0)$, we further find

$$\bar{u}_2(x, y, \beta_2, \beta'_1) = \bar{x}_4 x^\beta \sum_{k=0}^n \frac{(-x)^k}{k!(k+\beta)} D_x^k \bar{u}_1(x, y, \beta_1, \beta'_1) + R_n, \quad (8b)$$

$$R_n = \frac{\bar{x}_4}{n!} \int_0^x (x-\xi)^{n+\beta} B_{\xi/(\xi-x)}(1+n, \beta) D_\xi^{n+1} \bar{u}_1(\xi, y, \beta_1, \beta'_1) d\xi.$$

In the limit, when $n \rightarrow \infty$, the sum (8a) in the class $L_\infty(0, x_0)$ of functions $\nu(x)$ may be replaced by the infinite series

$$\bar{u}_2(x, y, \beta_2, \beta'_1) = \bar{x}_4 \Gamma(\beta) x^\beta \gamma^*(\beta, \delta_x) \bar{u}_1(x, y, \beta_1, \beta'_1), \quad (8c)$$

where $\beta_1 > \beta_2 \geq 0$; $\gamma(p, z) = \Gamma(p) z^p \gamma^*(p, z)$ is Euler's incomplete gamma function.

Theorem 4. Let $\tau(x) = P(x)\nu(x)$, $\beta_1 > 0$, $\beta'_1 > 0$, $\beta_2 \geq 0$, $\beta'_2 \geq 0$,

$$\mu_1 \Gamma(\beta_1) \Gamma(\beta'_1) \Gamma(\beta'_2) \Gamma(1 - \beta_2 - \beta'_2) (2 - a_1 - a'_1)^{\beta_1 + \beta'_1} = 2 \Gamma(1 - \beta_2) \Gamma(\beta_1 + \beta'_1),$$

and let the function M_1 be defined by the expression

$$M_1 = \mu_1(y-x)^{1-\beta_2-\beta'_2} \int_{\xi}^x P(t)(x-t)^{\beta'_2-1}(y-t)^{\beta_2+\beta'_1-1}(t-\xi)^{\beta_1-1} dt.$$

Then $\bar{u}_1(x, y, \beta_1, \beta'_1)$ is transformed into $u_2(x, y, \beta_2, \beta'_2)$ by the relation

$$u_2(x, y, \beta_2, \beta'_2) = \int_0^x M_1(\xi) D_{\xi} \bar{u}_1(\xi, y, \beta_1, \beta'_1) d\xi. \quad (9a)$$

In particular, if $P(x) = 1$,

$$M_1 = \bar{\mu}_1(y-x)^{-\beta'}(x-\xi)^{\beta_1+\beta'_2-1} F(1-\beta'_1-\beta_2, \beta'_2, \beta_1+\beta'_2; \omega),$$

$$\bar{\mu}_1 \Gamma(\beta_1 + \beta'_2) = \mu_1 \Gamma(\beta_1) \Gamma(\beta'_2).$$

This kernel becomes elementary for $\beta_2 = 1 - \beta'_1$, when $\mu_0 = \bar{\mu}_1|_{\beta_2=1-\beta'_1}$,

$$u_2(x, y, 1 - \beta'_1, \beta'_2) = \mu_0(y-x)^{-\beta'} \int_0^x (x-\xi)^{\beta_1+\beta'_2-1} D_{\xi} \bar{u}_1(\xi, y, \beta_1, \beta'_1) d\xi. \quad (9b)$$

Solving the integral equations (9) with respect to \bar{u} and assuming $\nu(x) = \bar{P}(x)\tau(x)$, $\bar{P}(x)P(x) = 1$, we obtain inverse relations. Thus, in the case $\beta_2 + \beta'_1 < 1$,

$$\bar{u}_2(x, y, \beta_2, \beta'_2) = \int_0^x M_2(\xi) u_1(\xi, y, \beta_1, \beta'_1) d\xi, \quad (10a)$$

$$M_2 = \mu_2(y-\xi)^{\beta_1+\beta'_1-1} D_{\xi} \int_{\xi}^x \bar{P}(t)(x-t)^{-\beta_2}(y-t)^{1-\beta_1-\beta'_2}(t-\xi)^{-\beta'_1} dt,$$

$$2\mu_2\Gamma(1-\beta_1)\Gamma(1-\beta'_1)\Gamma(1-\beta_2)\Gamma(\beta_2+\beta'_2) = -\Gamma(\beta'_2)\Gamma(1-\beta_1-\beta'_1)(2-a_2-a'_2)^{\beta_2+\beta'_2}.$$

For $\beta_2 + \beta'_1 = 1$, in the nonintegral term of the analogous relation formula there appears, as in (4), the function $P(x)u_1$. Now the value $P(x) = 1$ corresponds to the kernel

$$M_2 = \bar{\mu}_2 (y-x)^{1-\beta_1-\beta_2'} (x-\xi)^{-\beta_1'-\beta_2} (y-\xi)^{\beta_1+\beta_1'-1} F(\beta_1+\beta_2'-1, 1-\beta_2, 1-\beta_1'-\beta_2; \omega);$$

$$\bar{\mu}_2 \Gamma(1-\beta_1'-\beta_2) = -\Gamma(1-\beta_2) \Gamma(1-\beta_1') \mu_2,$$

which gives, for $\beta_2 = 1 - \beta_1$, $\beta_1' \geq 0$, $\beta_2 \geq 0$, $\beta_1' + \beta_2 < 1$, $\mu_3 = \bar{\mu}_2|_{\beta_2=1-\beta_1}$,

$$\bar{u}_2(x, y, \beta_2, 1 - \beta_1) = \mu_3 \int_0^x (x-\xi)^{-\beta_1'-\beta_2} (y-\xi)^{\beta_1+\beta_1'-1} u_1(\xi, y, \beta_1, \beta_1') d\xi. \quad (10b)$$

From the equalities (9b), (10b), under sufficient smoothness of the functions $\tau(x)$ and $\nu(x)$, expansions of the form (5b), (6), (8b), (8c) may be obtained.

More complicated forms are exhibited by similar representations for the other transformation operators found. Namely, introduce the notation

$$X_\beta^{\beta'} = k_1 \rho^{\beta'} \Phi_1(\beta', 1 - \beta, 1 + \beta'; -\rho, -\delta_x);$$

$$\bar{X}_\beta^{\beta'} = k_2 x^{1-\beta-\beta'} \rho^\beta \Phi_1(1 - \beta, \beta', 2 - \beta; -\rho, -\delta_x);$$

$$k_1 \Gamma(1 + \beta') \Gamma(1 - \beta - \beta') = \Gamma(1 - \beta),$$

$$2k_2 \Gamma(2 - \beta) \Gamma(\beta + \beta') = \Gamma(\beta') (2 - a - a')^{\beta+\beta'},$$

and consider the inverse operators

$$(X_\beta^{\beta'})^{-1} = k_3 (y-x)^{1-\beta} D_x [x^\beta \rho^{1-\beta-\beta'} \Phi_1(1 - \beta', 1 - \beta - \beta', 2 - \beta'; -\rho, -\delta_x)],$$

$$(\bar{X}_\beta^{\beta'})^{-1} = k_4 (y-x)^\beta D_x [x^\beta \gamma(\beta, \delta_x)],$$

where

$$k_3 \Gamma(1 - \beta) \Gamma(2 - \beta') = \Gamma(1 - \beta - \beta'),$$

$$k_4 \Gamma(\beta') (2 - a - a')^{\beta+\beta'} = 2\Gamma(\beta + \beta').$$

Then, assuming $\tau(x)$ and $\nu(x) \in L_\infty(0, x_0)$, we obtain, for example,

$$u_2 = X_{\beta_2}^{\beta_2'} \left[P(x)(X_{\beta_1}^{\beta_1'})^{-1} u_1 \right], \quad \bar{u}_2 = \bar{X}_{\beta_2}^{\beta_2'} \left[P(x)(\bar{X}_{\beta_1}^{\beta_1'})^{-1} \bar{u}_1 \right].$$

It is of interest, as in ⁽¹⁾, to investigate the confluent cases of equation (1). Let us replace x in (1) and (2a) by εx , and then, putting $\beta = -\alpha/\varepsilon$, pass to the limit as $\varepsilon \rightarrow 0$. This will give, for the function $z(x, y, \beta') = \lim_{\varepsilon \rightarrow 0} u(\varepsilon x, y, -\alpha/\varepsilon, \beta')$,

$$yz_{xy} + \beta' z_x + \alpha z_y = 0; \quad z(0, y) = 0, \quad z(x, 0) = \tau(x), \quad \tau(0) = 0. \quad (11)$$

Carrying out the same passage to the limit in the equalities (5), (6) and taking into account that $\lim_{z \rightarrow \infty} [z^\alpha \Gamma(z)]^{-1} \Gamma(z + a) = 1$, we find connection formulas ⁽²⁾ for solutions of the singular Goursat problem (11):

$$z_2(x, y, \beta_2') = \left(\frac{\alpha x}{y} \right)^{\beta'} \gamma^* \left(\beta', \frac{\alpha x}{y} + \delta_x \right) z_1(x, y, \beta_1') \quad (\beta' = \beta_2' - \beta_1' > 0),$$

$$z_2(x, y, \beta_2') = \frac{1}{\Gamma(\beta')} \left(\frac{y}{\alpha} \right)^{-\beta'} \int_0^x (x - \xi)^{\beta' - 1} \exp \left[-\frac{\alpha(x - \xi)}{y} \right] z_1(\xi, y, \beta_1') d\xi.$$

The functions u and z can also be connected in another way. For example, the relation

$$u_2(x, y, \beta_2, \beta_2') = \lambda_0 \left(\frac{y}{\alpha} \right)^{\beta_1'} (y - x)^{1 - \beta_2 - \beta_2'} \int_0^x Q(\xi) z_1(\xi, y, \beta_1') d\xi, \quad (12)$$

holds, where $\tau_2(x) = P(x)\tau_1(x)$,

$$\lambda_0 \Gamma(\beta_2') \Gamma(1 - \beta_1') \Gamma(1 - \beta_2 - \beta_2') = -\Gamma(1 - \beta_2),$$

$$\beta_2' > \beta_1' \geq 0, \quad Q = e^{\alpha\xi/y} D_\xi \int_\xi^x (x - t)^{\beta_2' - 1} (y - t)^{\beta_2 - 1} (t - \xi)^{-\beta_1'} P(t) e^{-\alpha t/y} dt.$$

For $\beta_2' = \beta_1'$, an additional term appears on the right-hand side of (12):

$$\frac{\Gamma(1 - \beta_2)}{\Gamma(1 - \beta_2 - \beta_1')} \left[\frac{y}{\alpha(y - x)} \right]^{\beta_1'} P(x) z_1;$$

if $P(x) = 1$, then $Q(\xi)$ reduces to the function Φ_1 . By similar equalities one also effects the inverse transformation of the integrals u_1 into the solutions z_2 . The connection formulas obtained, as a result of a suitable particular choice of

the values of $\tau(x)$, $\nu(x)$, and $P(x)$, make it possible to compute a number of integrals with known or new special functions.

Also worthy of attention are the initial problems considered for the functions

$$(u-x)^{-\beta'} u\left(x, x + \frac{1}{y-x}\right) \quad \text{and} \quad y^{-\beta'} z\left(x, \frac{1}{y}\right).$$

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- ¹ P. Humbert, *Proc. Roy. Soc. Edinburgh*, **41**, Part I, 73 (1920-1921).
² M. B. Kapilevich, *DAN*, **130**, No. 3 (1960).

Note: Figure translations are in progress. See original paper for figures.

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