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Abstract

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MATHEMATICS

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ON THE INDUCTION OF A CONNECTION OF CONSTANT CURVATURE IN ASSOCIATED CENTRO-PROJECTIVE SPACES OF A LOCALLY PROJECTIVE MANIFOLD

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1. Let, in a locally projective manifold, i.e. in a manifold defined by nondegenerate fractional-linear transformations of local coordinate systems,

$$x^{i'} = \frac{a_i^{i'} x^i}{b_i x^i + 1} + c^{i'} \qquad (a_i^{i'},\, b_i,\, c^{i'} = \text{const}), \tag{1}$$

there be given a principal relative scalar $a = a(x)$ of weight one. Then the object

$$\gamma^p_{jk}(x) = \frac{1}{(n+1)} \delta^p_j \frac{\partial \ln a(x)}{\partial x^k} + \frac{1}{(n+1)} \delta^p_k \frac{\partial \ln a(x)}{\partial x^j} \tag{2}$$

may be taken as the object of an affine connection ([1]), while the curvature tensor and the Ricci tensor, as direct computations show, take, respectively, the form

$$R^h_{kij}(x) = \frac{1}{(1-n)} \left(\delta^h_i R_{jk} - \delta^h_j R_{ik}\right); \tag{3}$$

$$R_{ij}(x) = \frac{1-n}{n+1} \left(\frac{1}{(n+1)} \frac{\partial \ln a}{\partial x^i} \frac{\partial \ln a}{\partial x^j} - \frac{\partial^2 \ln a}{\partial x^i \partial x^j} \right) = (1-n)\, a^{\frac{1}{n+1}} \frac{\partial^2 a^{-\frac{1}{n+1}}}{\partial x^i \partial x^j}. \tag{4}$$

In what follows we shall assume that

$$\det \|R_{ij}(x)\| \neq 0. \tag{5}$$

2. The principal relative scalar in each local centro-projective space $\{P^n\}$ ([2]) determines the quantity

$$\alpha(x,u)=\frac{a(x)}{\left(\frac{a^{\frac{1}{n+1}}}{2}\frac{\partial^2 a^{-\frac{2}{n+1}}}{\partial x^i\partial x^j}u^iu^j-\frac{2}{(n+1)}\frac{\partial\ln a}{\partial x^i}u^i+1\right)^{\frac{n+1}{2}}}, \tag{6}$$

which is transformed under fractional-linear transformations

$$u^{i'}=\frac{\dfrac{\partial x^{i'}}{\partial x^i}u^i}{-\dfrac{1}{(n+1)}\dfrac{\partial\ln\det\|\partial x^{r'}/\partial x^r\|}{\partial x^q}u^q+1},$$

induced by the transformations (1) in $\{P^n\}$, according to the law of a relative scalar of weight one and satisfying the conditions

$$\alpha(x,u)\big|_{u=0}=a(x),\qquad \frac{\partial\alpha(x,u)}{\partial u^k}\bigg|_{u=0}=\frac{\partial a(x)}{\partial x^k},\qquad \frac{\partial\alpha(x,u)}{\partial u^k\partial u^l}\bigg|_{u=0}=\frac{\partial^2 a(x)}{\partial x^k\partial x^l}. \tag{7}$$

From α one can construct in $\{P^n\}$ the connection

$$\Gamma^p_{jk}(x,u)=$$

$$=\frac{\delta^p_j\left(\frac{1}{(n+1)}\frac{\partial\ln a}{\partial x^k}-\frac{a^{\frac{2}{n+1}}}{2}\frac{\partial^2 a^{-\frac{2}{n+1}}}{\partial x^k\partial x^q}u^q\right)+\delta^p_k\left(\frac{1}{(n+1)}\frac{\partial\ln a}{\partial x^j}-\frac{a^{\frac{2}{n+1}}}{2}\frac{\partial^2 a^{-\frac{2}{n+1}}}{\partial x^j\partial x^q}u^q\right)}{\frac{a^{\frac{2}{n+1}}}{2}\frac{\partial^2 a^{-\frac{2}{n+1}}}{\partial x^q\partial x^l}u^qu^l-\frac{2}{(n+1)}\frac{\partial\ln a}{\partial x^q}u^q+1}. \tag{8}$$

3. Theorem. In order that the connection defined in $\{P^n\}$ by the object

$$\gamma^p_{jk}(u)=\frac{1}{(n+1)}\delta^p_j\frac{\partial\ln\alpha}{\partial u^k}+\frac{1}{(n+1)}\delta^p_k\frac{\partial\ln\alpha}{\partial u^j}, \tag{9}$$

be a connection of constant curvature $K\neq 0$, it is necessary and sufficient that it be Riemannian; and this is possible if and only if

$$\alpha=\frac{c}{c_{ij}u^iu^j+2c_iu^i+1},\qquad \det\|c_ic_j-c_{ij}\|\neq 0. \tag{10}$$

Moreover, under the natural requirement

$$\alpha^2=\det\|g_{ij}\| \tag{11}$$

the curvature K and the metric tensor itself ($g_{ij} = g_{ji}$) are determined up to factors ε (respectively $\bar{\varepsilon}$), equal to n-th roots of unity, and

$$c^2 = \frac{1}{K^n} \det \|c_i c_j - c_{ij}\|.$$

Proof. The requirement of covariant constancy of the tensor g_{ij} with respect to the connection (9) leads to the system

$$\frac{\partial g_{ij}}{\partial u^k} = \frac{1}{(n+1)} \left(g_{ik} \frac{\partial \ln \alpha}{\partial u^j} + 2 g_{ij} \frac{\partial \ln \alpha}{\partial u^k} + g_{kj} \frac{\partial \ln \alpha}{\partial u^i} \right), \tag{12}$$

whose integrability conditions have the form

$$g_{ik} \left(\frac{1}{(n+1)} \frac{\partial \ln \alpha}{\partial u^j} \frac{\partial \ln \alpha}{\partial u^l} - \frac{\partial^2 \ln \alpha}{\partial u^j \partial u^l} \right) + g_{kj} \left(\frac{1}{(n+1)} \frac{\partial \ln \alpha}{\partial u^i} \frac{\partial \ln \alpha}{\partial u^l} - \frac{\partial^2 \ln \alpha}{\partial u^i \partial u^l} \right)$$
$$- g_{il} \left(\frac{1}{(n+1)} \frac{\partial \ln \alpha}{\partial u^j} \frac{\partial \ln \alpha}{\partial u^k} - \frac{\partial^2 \ln \alpha}{\partial u^j \partial u^k} \right) - g_{lj} \left(\frac{1}{(n+1)} \frac{\partial \ln \alpha}{\partial u^i} \frac{\partial \ln \alpha}{\partial u^k} - \frac{\partial^2 \ln \alpha}{\partial u^i \partial u^k} \right) = 0. \tag{13}$$

Hence, in particular, for $i = j$ we have

$$g_{ik} \left(\frac{1}{(n+1)} \frac{\partial \ln \alpha}{\partial u^i} \frac{\partial \ln \alpha}{\partial u^l} - \frac{\partial^2 \ln \alpha}{\partial u^i \partial u^l} \right) = g_{il} \left(\frac{1}{(n+1)} \frac{\partial \ln \alpha}{\partial u^i} \frac{\partial \ln \alpha}{\partial u^k} - \frac{\partial^2 \ln \alpha}{\partial u^i \partial u^k} \right).$$

It is now not difficult to see that system (13) is equivalent to

$$\frac{1}{(n+1)} \frac{\partial \ln \alpha}{\partial u^i} \frac{\partial \ln \alpha}{\partial u^j} - \frac{\partial^2 \ln \alpha}{\partial u^i \partial u^j} = -(n+1) K \cdot g_{ij}. \tag{14}$$

Substituting into (12) the value of g_{ij} from (14), we obtain

$$-\frac{\partial \ln K}{\partial u^k} \left(\frac{1}{(n+1)} \frac{\partial \ln \alpha}{\partial u^i} \frac{\partial \ln \alpha}{\partial u^j} - \frac{\partial^2 \ln \alpha}{\partial u^i \partial u^j} \right) =$$

$$= \frac{4}{(n+1)^2} \frac{\partial \ln \alpha}{\partial u^i} \frac{\partial \ln \alpha}{\partial u^j} \frac{\partial \ln \alpha}{\partial u^k} - \frac{2}{(n+1)} \frac{\partial \ln \alpha}{\partial u^{(i}} \frac{\partial^2 \ln \alpha}{\partial u^j \partial u^{k)}} + \frac{\partial^3 \ln \alpha}{\partial u^i \partial u^j \partial u^k}. \tag{15}$$

However, in fact equations (15) can be written in the simpler form

$$\frac{\partial^3 \ln \alpha}{\partial u^i \partial u^j \partial u^k} - \frac{2}{(n+1)} \frac{\partial \ln \alpha}{\partial u^{(i}} \frac{\partial^2 \ln \alpha}{\partial u^i \partial u^{j)}} + \frac{4}{(n+1)^2} \frac{\partial \ln \alpha}{\partial u^i} \frac{\partial \ln \alpha}{\partial u^j} \frac{\partial \ln \alpha}{\partial u^k}, \tag{16}$$

for $K = \text{const}$. Indeed, by formulas (3) and (4), written for the connection (9), we have

$$R_{ij}(u) = (n-1)K \cdot g_{ij}, \qquad R_{lk,ij}(u) = K \cdot (g_{lj}g_{ki} - g_{li}g_{kj}). \tag{17}$$

Thus, for $n > 2$, $K = \text{const}$.

Further, for $n = 2$, by virtue of the symmetry of the right-hand side of (15) with respect to any pair of indices, the equalities

$$\left(\frac{1}{(n+1)}\frac{\partial \ln\alpha}{\partial u^1}\frac{\partial \ln\alpha}{\partial u^1} - \frac{\partial^2 \ln\alpha}{(\partial u^1)^2}\right)\frac{\partial \ln K}{\partial u^2} - \left(\frac{1}{(n+1)}\frac{\partial \ln\alpha}{\partial u^1}\frac{\partial \ln\alpha}{\partial u^2} - \frac{\partial^2 \ln\alpha}{\partial u^1 \partial u^2}\right)\frac{\partial \ln K}{\partial u^1} = 0,$$

$$-\left(\frac{1}{(n+1)}\frac{\partial \ln\alpha}{\partial u^2}\frac{\partial \ln\alpha}{\partial u^1} - \frac{\partial^2 \ln\alpha}{\partial u^2 \partial u^1}\right)\frac{\partial \ln K}{\partial u^2} - \left(\frac{1}{(n+1)}\frac{\partial \ln\alpha}{\partial u^2}\frac{\partial \ln\alpha}{\partial u^2} - \frac{\partial^2 \ln\alpha}{(\partial u^2)^2}\right)\frac{\partial \ln K}{\partial u^1} = 0$$

must hold. Consequently, $K = \text{const}$, since $\det \|R_{ij}(u)\| \neq 0$.

Solving now system (16), which can be rewritten in the form

$$\frac{\partial\left(\frac{\partial \ln\alpha}{\partial u^i}\,\alpha^{-\frac{2}{n+1}}\,\frac{\partial \ln\alpha}{\partial u^j}\,\alpha^{-\frac{2}{n+1}}\right)}{\partial u^k} + \frac{\partial\left(\frac{\partial^2 \ln\alpha}{\partial u^i \partial u^j}\,\alpha^{-\frac{2}{n+1}}\right)}{\partial u^k} = 0$$

or

$$\frac{\partial^2 \alpha^{-\frac{2}{(n+1)}}}{\partial u^i \partial u^j} = \tilde{c}_{ij},$$

where $\tilde{c}_{ij} = \text{const}$ and $\tilde{c}_{ij} = \tilde{c}_{ji}$, we obtain

$$\alpha = \frac{c}{\left(c_{ij}u^i u^j + 2c_i u^i + 1\right)^{\frac{n+1}{2}}}. \tag{18}$$

Further, from (17) we have

$$\det \|R_{ij}(u)\| = (n-1)^n \cdot K^n \cdot \det \|g_{ij}\|,$$

therefore, by virtue of conditions (11) and (10), we obtain

$$c^2 = \frac{1}{K^n} \cdot \det \|c_i c_j - c_{ij}\|.$$

It is also clear from this that the curvature K and the metric tensor g_{ij} are determined up to factors ε (respectively ε).

4. The connection (8) is obtained by substituting (6) into (9), and since (6) has the same structure as (18), while, by virtue of (5),

$$\det \|R_{ij}(x,u)\| = (n-1)^n K^n(x) \det \|g_{ij}(x,u)\| \neq 0, \tag{19}$$

then (8) defines in each $\{P^n\}$ a connection of constant curvature. Further, by virtue of (7),

$$\begin{aligned} \Gamma^p_{jk}(x,u)\big|_{u=0} &= \gamma^p_{jk}(x), & \frac{\partial \Gamma^p_{jk}(x,u)}{\partial u^l}\bigg|_{u=0} &= \frac{\partial \gamma^p_{jk}(x)}{\partial x^l}, \\ R^h{}_{kij}(x,u)\big|_{u=0} &= R^h{}_{kij}(x), & R_{ij}(x,u)\big|_{u=0} &= R_{ij}(x). \end{aligned} \tag{20}$$

Finally, under the natural requirement

$$\alpha^2(x,u) = \det \|g_{ij}(x,u)\|$$

from (19), taking (20) into account, we have

$$K(x) = \frac{\varepsilon}{(n-1)} \sqrt[n]{\frac{\det \|R_{ij}(x)\|}{a^2(x)}}, \tag{21}$$

and, consequently, from

$$R_{ij}(x,u) = (n-1)K(x) g_{ij}(x,u)$$

we obtain

$$g_{ij}(x,u) = \bar{\varepsilon} R_{ij}(x,u) \sqrt[n]{\frac{a^2(x)}{\det \|R_{ij}(x)\|}},$$

and for $u = 0$

$$g_{ij} = (x) \bar{\varepsilon} R_{ij}(x) \sqrt[n]{\frac{a^2(x)}{\det \|R_{ij}(x)\|}}. \tag{22}$$

Let us note in conclusion that the curvature (21) is a function of the point of the locally projective manifold and, consequently, when $K \neq \text{const}$, the covariant derivative of the tensor (22) with respect to the connection (2) is different from zero.

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References Cited

[1] V. G. Lemlein, DAN, **128**, No. 4 (1959).
[2] V. G. Lemlein, DAN, **129**, No. 2 (1959).

Note: Figure translations are in progress. See original paper for figures.