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Mathematics

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Abstract

Full Text

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ON THE SUMMATION OF ORTHOGONAL SERIES BY LINEAR METHODS

Let us take a matrix

$$B = \|b_{in}\| \tag{1}$$

with elements b_{in}, $i, n = 1, 2, ...$, and some numerical series

$$\sum_{n=1}^{\infty} u_n. \tag{2}$$

Denote by S_n the partial sums of the series (2), i.e., put

$$S_n = \sum_{k=1}^{n} u_k. \tag{3}$$

As is known, the series (2) is called summable to the value S by the linear method determined by the matrix B if, first, the series

$$T_i = \sum_{n=1}^{\infty} b_{in} S_n \tag{4}$$

converges for every i, and, second,

$$\lim_{i\to\infty} T_i = S. \tag{5}$$

A linear method is called regular if every series converging to a finite sum S is summable by the given method to the same sum S.

Necessary and sufficient conditions are known for a linear method to be regular.

A linear method determined by a matrix B with real elements is called completely regular if, first, it is regular in the usual sense and if, second, every real series converging to $+\infty$ or to $-\infty$ is summable by the given method to infinity of the same sign.

Examples of completely regular linear methods are the Cesàro methods (C, α) of positive order α.

Let us take some orthonormal system $\{\varphi_n(x)\}$ on the interval $[a, b]$. The series

$$\sum_{n=1}^{\infty} c_n \varphi_n(x) \tag{6}$$

is called an orthogonal series from L^2 if it is the Fourier series of a square-integrable function with respect to the system $\{\varphi_n(x)\}$, i.e., if

$$\sum_{n=1}^{\infty} c_n^2 < +\infty. \tag{7}$$

The following theorem is known:

In order that the orthogonal series (6) from L^2 be summable by the Cesàro method of positive order almost everywhere on $[a, b]$, it is necessary and sufficient that

$$\lim_{u \to \infty} S_{2^u}(x) \tag{8}$$

exist and have a finite value almost everywhere on $[a, b]$, where $u = 1, 2, \ldots$ and $S_n(x)$ is the partial sum of the series (6).

This theorem was proved by Kaczmarz for the method $(C, 1)$ and by Zygmund for the method (C, α) for any $\alpha > 0$*.

The question arises whether the preceding theorem will remain true for any linear regular summability method if, in its formulation, the sequence of numbers 2^u, $u = 1, 2, \ldots$, is replaced by some other increasing sequence of natural numbers n_u, $u = 1, 2, \ldots$, depending in general on the summability method. The answer to this question is negative. In order to formulate the result, we introduce two definitions.

Definition 1. Let two linear methods A and B and an ON system $\{\varphi_n(x)\}$, defined on the interval $[a, b]$, be given. We shall say that the methods A and B are **equivalent in the space** L^2 for the system $\{\varphi_n(x)\}$ if every Fourier series of a function from $L^2[a, b]$ either is summed simultaneously by the methods A and B to one and the same sum almost everywhere on $[a, b]$, or is not summed simultaneously by these methods almost everywhere on the same interval.

Definition 2. Let an increasing sequence of natural numbers n_u, $u = 1, 2, \ldots$, be given. We shall say that the numerical series (2) is **summed by the method** $T[n_u]$ **to the sum** S, if

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$$\lim_{u\to\infty} S_{n_u} = S,$$

where S_n is defined by equality (3).

Kaczmarz proved the following theorem:

*If condition (7) is satisfied and if the orthogonal series (6) is summed by some linear regular method B almost everywhere on $[a,b]$, then there exists an increasing sequence of natural numbers n_u, $u = 1, 2, \dots$, depending only on the method B, such that the series (6) is summed by the method $T[n_u]$ almost everywhere on $[a,b]$**.*

However, in this theorem the method $T[n_u]$ is not necessarily equivalent to the method B in the sense of Definition 1; namely, one can prove the following theorem:

Theorem A. *There exists a linear fully regular method B and an ON system of functions $\varphi_n(x)$, $n = 1, 2, \dots$, defined and bounded in their totality on $[0,1]$, such that for every increasing sequence of natural numbers n_u, $u = 1, 2, \dots$, the method B is not equivalent to the method $T[n_u]$ in the space L^2 for the given system of functions.*

The elements b_{ni}, $n, i = 1, 2, \dots$, of the matrix corresponding to the method B, which is discussed in Theorem A, are defined as follows.

First of all, we define the numbers ν_r by the equality

$$\nu_r = 2^{r^{16}} \quad (r = 1, 2, \dots).$$

* (^{1}), Theorem 1.1.8, p. 219, and Theorem 5.8.5, p. 222.
** (^{1}), Theorem 5.7.4, p. 214.

Next put

$$N_0 = 0, \qquad N_r = 2\sum_{\rho=1}^{r} \nu_\rho^2, \qquad N'_r = N_{r-1} + \nu_r^2 \quad (r = 1, 2, \dots)$$

and, for each $i = 1, 2, \dots$, define the natural number r_i from the condition

$$N_{r_i - 1} < i \le N_{r_i} \qquad (i = 1, 2, \dots).$$

It is clear that

$$N_{r_i - 1} < N'_{r_i} < N_{r_i} \qquad (i = 1, 2, \dots).$$

We now define the quantities b_{in} as follows:

1) If

$$N_{r_i-1} < i < N'_{r_i},$$

then we put

$$b_{ii} = \eta_i, \qquad b_{i,i+\nu^2_{r_i}} = \eta_i, \qquad b_{i,N_{r_i}} = 1,$$

$$b_{in} = 0 \qquad (n = 1, 2, \dots;\ n \neq i;\ n \neq i + \nu^2_{r_i};\ n \neq N_{r_i}),$$

where

$$\eta_i = \frac{1}{\sqrt[4]{\lg \nu_{r_i}}}.$$

2) If

$$i = N'_{r_i},$$

then we put

$$b_{ii} = \eta_i, \qquad b_{i,N_{r_i}} = 1 + \eta_i,$$

$$b_{in} = 0, \qquad (n = 1, 2, \dots;\ n \neq i;\ n \neq N_{r_i}).$$

3) If

$$N'_{r_i} < i \leq N_{r_i},$$

then we put

$$b_{i,N_{r_i}} = 1, \qquad b_{in} = 0 \qquad (n = 1, 2, \dots;\ n \neq N_{r_i}).$$

It is easy to show that the method B, defined by the matrix $\|b_{in}\|$, is a completely regular method. Moreover, on $[0, 1]$ one can define such an ON system of functions $\varphi_n(x)$, $n = 1, 2, \dots$, bounded in their totality, that for the method B defined above and for this system of functions theorem A will be valid. Thus theorem A will be proved.

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CITED LITERATURE

1. S. Kaczmarz, H. Steinhaus, *Theory of Orthogonal Series*, translated by R. S. Guter and P. L. Ulyanov, IL, 1958.

Note: Figure translations are in progress. See original paper for figures.

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