MATHEMATICS

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1960

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Abstract

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MATHEMATICS

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ON FINDING THE GROUP OF A LINEAR DIFFERENTIAL EQUATION OF SECOND ORDER

(Presented by Academician I. N. Vekua on 28 XII 1959)

In the present paper we report some results connected with the problem of finding the group of transformations preserving a given linear differential equation with partial derivatives of second order for an arbitrary number n of independent variables ($n > 1$). After deriving the determining equations of the infinitesimal transformations, we establish an invariant form of these equations, containing only the Riemannian space associated with the given equation, a vector of this space, and an invariant. A final solution of the problem has not yet been obtained; here we give only some general facts.

Let us note that the original general formulation of such problems—the problems of group classification of systems of differential equations—belongs to S. Lie ([1]). In particular, he obtained a classification of linear equations of second order for the case $n = 2$, though in noninvariant form. This formulation differs from that recently proposed in the work of I. M. Gelfand ([2]), where the unknowns are equations admitting a given group.

1. Let $u = u(x^1, x^2, \dots, x^n)$, $u_i = \partial u/\partial x^i$, $u_{ij} = \partial^2 u/\partial x^i \partial x^j$. We introduce notation for systems of values of the quantities indicated in parentheses: $x = (x^1, x^2, \dots, x^n)$, $v = (x, u)$, $p = (v, u_1, u_2, \dots, u_n)$, $q = (p, u_{11}, u_{12}, \dots, u_{nn})$. Consider the equation

$$F(q) \equiv a^{ij} u_{ij} + b^i u_i + cu = 0 \quad (a^{ij} = a^{ji}), \tag{1}$$

where a^{ij}, b^i, c are prescribed analytic functions of x, and the tensor convention of summation over a repeated index is used (as below) (all indices take the values $1, 2, \dots, n$).

The problem is posed as follows: to find all transformations of the form

$$p' = f(p), \tag{2}$$

belonging to certain one-parameter Lie groups and preserving equation (1) in the sense that $F(q') = 0$ whenever $F(q) = 0$.

These transformations generate a local Lie group ([3]) of transformations, denoted below by G_F. In this case one says that equation (1) admits the transformations (2), or, respectively, the group G_F.

We observe that G_F always contains transformations distinct from the identity. Such transformations are

$$x' = x, \qquad u' = \alpha u + u_0(x), \tag{3}$$

where α is an arbitrary parameter, and $u_0(x)$ is any fixed solution of equation (1). The set of all transformations of the form (3) is itself a group and therefore a subgroup of the group G_F. We shall denote this subgroup by T.

2. A one-parameter group of transformations of the form (2) is completely characterized by its infinitesimal operator

$$Y = \xi^i(p)\frac{\partial}{\partial x^i} + \eta(p)\frac{\partial}{\partial u} + \zeta_i(p)\frac{\partial}{\partial u_i}, \tag{4}$$

which can be prolonged to the second derivatives u_{ij} and yield an operator acting on functions of q:

$$\widetilde{Y} = \xi^i\frac{\partial}{\partial x^i} + \eta\frac{\partial}{\partial u} + \zeta_i\frac{\partial}{\partial u_i} + \omega_{ij}\frac{\partial}{\partial u_{ij}}. \tag{5}$$

Here the quantities ζ_i and ω_{ij} are computed by the known prolongation formulas

$$\zeta_i = D_i(\eta) - u_{jD}i(\xi^j), \qquad \omega_{ij} = D_i(\zeta_j) - u_{jl}D_i(\xi^l), \tag{6}$$

where the differential operations D_i are defined by the formula

$$D_i = \frac{\partial}{\partial x^i} + u_i\frac{\partial}{\partial u} + u_{ij}\frac{\partial}{\partial u_j} \qquad (i = 1, 2, \dots, n). \tag{7}$$

In terms of the operator (4), the problem posed is formulated as follows: find all operators Y such that $\widetilde{Y}F(q) = 0$ whenever $F(q) = 0$. In more detailed notation this invariance condition has the form

$$\left[a^{ij}\omega_{ij} + b^i\zeta_i + c\eta + \xi^k\left(\frac{\partial a^{ij}}{\partial x^k}u_{ij} + \frac{\partial b^i}{\partial x^k}u_i + \frac{\partial c}{\partial x^k}u\right)\right]_{F(q)=0} = 0. \tag{8}$$

According to the statement of the problem, the quantities ζ_i must not depend on u_{ij}. Therefore the aggregate of the terms in the expressions for ζ_i according to formulas (6), (7) that contain u_{ij} must vanish under the condition $F(q) = 0$.

For $n > 1$ the following holds:

Lemma 1. In order that the quantities ζ_i not depend on u_{ij} when $F(q) = 0$, it is necessary and sufficient that there exist a function $\varphi = \varphi(p)$ such that

$$\xi^i = -\frac{\partial \varphi}{\partial u_i}, \qquad \eta = \varphi - u_i \frac{\partial \varphi}{\partial u_i}, \qquad \zeta_i = \frac{\partial \varphi}{\partial x^i} + u_i \frac{\partial \varphi}{\partial u} \qquad (i = 1, 2, \dots, n). \quad (9)$$

3. We shall call an equation $F'(q') = 0$ **equivalent** to equation (1) if it is obtained from (1) as a result of a change of variables (substitution) of the form

$$x' = x'(x), \qquad u(x) = \psi(x) u'(x') \qquad (10)$$

and subsequent division by $\psi(x)$. If in (10) $x' \equiv x$, then the corresponding equation $F'(q') = 0$ will be called **equivalent in the function** to equation (1).

Under the change of variables (10), the group G_F is transformed into a similar group $G_{F'}$. Therefore the problem of finding G_F is equivalent to the problem of finding any $G_{F'}$.

Lemma 2. Except for the case when (1) is equivalent to the equation $u_{12} = 0$, the function $\varphi(p)$ is linear in the variables $u, u_1, u_2, \dots, u_n$.

In what follows we shall set aside the case of the equation $u_{12} = 0$. Then from Lemma 2, in view of (9), it follows that $\xi^i = \xi^i(x)$ $(i = 1, 2, \dots, n)$, $\eta = \sigma(x)u + \tau(x)$. Therefore the transformations G_F can be represented as transformations of the form $v' = f(v)$. Moreover, as a more detailed consideration shows, the function $\tau(x)$ must be a solution of equation (1). Hence, in conjunction with the analytic sign of a normal divisor [4], it follows

Lemma 3. The subgroup T is a normal divisor in G_F. The transformations of the factor group G_F/T are characterized by infinitesimal operators of the form

$$X = \xi^i(x) \frac{\partial}{\partial x^i} + \sigma(x) u \frac{\partial}{\partial u}, \qquad (11)$$

where $\sigma(x)$ is determined up to a constant summand.

By Lemmas 2 and 3 and formulas (6), (7), (9), condition (8), whose left-hand side turns out to be a linear function of the quantities u_{ij}, "splits" and gives rise to the following system of equations:

$$a^{ik} \frac{\partial \xi^j}{\partial x^k} + a^{jk} \frac{\partial \xi^i}{\partial x^k} - \frac{\partial a^{ij}}{\partial x^k} \xi^k = \mu a^{ij} \quad (i, j = 1, 2, \dots, n); \qquad (12)$$

$$2a^{ik}\frac{\partial\sigma}{\partial x^k} = a^{kl}\frac{\partial^2\xi^i}{\partial x^k\partial x^l} + b^k\frac{\partial\xi^i}{\partial x^k} - \frac{\partial b^i}{\partial x^k}\xi^k - \mu b^i \quad (i = 1, 2, \dots, n); \tag{13}$$

$$a^{ij}\frac{\partial^2\sigma}{\partial x^i\partial x^j} + b^i\frac{\partial\sigma}{\partial x^i} = -\frac{\partial c}{\partial x^i}\xi^i - \mu c, \tag{14}$$

where $\mu = \mu(x)$ is an auxiliary function to be eliminated from this system. Equations (12)–(14) are the **determining equations** of the infinitesimal transformations of the group G_F/T. With the aid of these equations the following theorem is easily proved (the converse of which is trivial):

Theorem 1. If equation (1) admits at least one (nonzero) operator (11), then it is equivalent to an equation all of whose coefficients do not depend on one of the coordinates.

4. In what follows we shall restrict ourselves to the assumption that the total rank of the matrix $\|a^{ij}\|$ is equal to n. Then, observing that under a change of independent variables (10) the coefficients a^{ij} transform as components of a contravariant tensor, we may introduce into consideration the covariant components of this tensor a_{ij} ($a_{ij}a^{jk} = \delta_i^k$). Thus with equation (1) there is associated a Riemannian space V_n with fundamental form $a_{ij}dx^i dx^j$. We shall proceed to find forms of the determining equations (12)–(14) that are invariant with respect to V_n.

Since equations (12) are nothing other than the determining equations of conformal transformations of V_n (5), the preceding considerations lead to the following theorem.

Theorem 2. The group G_F/T of a nondegenerate equation (1) is a subgroup of the group of conformal transformations of the Riemannian space V_n associated with this equation.

It follows from this, in particular, that for $n > 2$ the group G_F/T has order not exceeding $(n+1)(n+2)/2$.

The coefficients b^i of equation (1) are not components of a vector. However, it is easy to verify that the quantities

$$a^i = b^i + a^{kl}\Gamma^i_{kl} \quad (i = 1, 2, \dots, n), \tag{15}$$

where Γ^i_{kl} are the Christoffel symbols of the second kind, computed for the associated V_n, already form a contravariant vector.

Equations (13) may be regarded as equations serving to determine the function $\sigma(x)$. If, by means of the covariant components a_i of the vector a^i, one forms the skew-symmetric tensor $K_{ij} = a_{i,j} - a_{j,i}$ (commas denote covariant derivatives with respect to the tensor a_{ij}), then the integrability conditions for equations (13) can be given the following invariant form:

$$(K_{il}\xi^l)_{,j} = (K_{jl}\xi^l)_{,i} \quad (i, j = 1, 2, \dots, n). \tag{16}$$

Equation (14), by eliminating σ with the aid of (13), is reduced to the following invariant form:

$$H_{,i}\xi^i + \mu H = 0, \tag{17}$$

where H is the invariant defined by the formula

$$H = -2c + a^i{}_{,i} + \frac{1}{2}a^i a_i + \frac{n-2}{2(n-1)}R, \tag{18}$$

where R is the scalar curvature of the associated V_n.

The analytic nature of the quantities K_{ij} and H is clarified by the following proposition:

Lemma 4. *In order that two equations of the form (1) with the same V_n be equivalent with respect to a function, it is necessary and sufficient that these equations have the same tensor K_{ij} and invariant H.*

Thus, the collection of quantities K_{ij} and H is a generalization of the Laplace invariants known for the case $n = 2$ ([6]).

5. If one regards the group of conformal transformations of V_n as an already studied object, then there next arises the problem of investigating the restrictions of geometric or group-theoretic character imposed on the group G_F/T by the additional conditions (16) and (17).

An important property of H is given by the following lemma:

Lemma 5. *In passing from equation (1) to the equation $g(x)F(q) = 0$, the invariant H is multiplied by $g(x)$.*

This property shows that it is necessary to distinguish the cases $H = 0$ and $H \neq 0$. We shall consider here only the latter.

If $H \neq 0$, then, by Lemma 5, without loss of generality one may take $H = 1$. Then from (17) it follows that $\mu = 0$, which proves the following theorem:

Theorem 3. *If $H \neq 0$, then G_F/T is a subgroup of the group of motions of some Riemannian space conformally associated with V_n.*

As a particular consequence of this we obtain that, for $n > 2$, the order of the group G_F/T of an equation for which $H \neq 0$ does not exceed $n(n+1)/2$.

Of interest is the question of the cases in which the group G_F/T may have the greatest possible order. We shall agree in general to call a **standard form** of equation (1), admitting G_F/T with the given properties, the simplest of the possible forms of this equation obtained by passing to an equivalent equation and multiplying by some factor.

Let us note that if, for $H \neq 0$, equation (1) admits G_F/T of maximal order, then V_n', associated with the equation $(1/H)F(q) = 0$, has constant curvature K_0 ([5]). Hence, after some computations, one obtains:

Theorem 4. *For $n > 2$, a standard form of equation (1) admitting G_F/T of order $n(n+1)/2$ is either* $(K_0 = 0)$

$$\Delta u + u = 0,$$

or $(K_0 \neq 0)$

$$\Delta u + \frac{n(n-2)}{(1+r^2)^2} u = 0,$$

where Δ is the Laplace operator, $r^2 = (x^1)^2 + (x^2)^2 + \cdots + (x^n)^2$.

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Received
23 XII 1959

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Note: Figure translations are in progress. See original paper for figures.