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Abstract

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INTEGRALS OVER HYPERPLANES OF BASIC AND GENERALIZED FUNCTIONS

1. To each rapidly decreasing function f in an n-dimensional real or complex affine space we associate its integrals over all possible hyperplanes. The resulting function, defined on the set of hyperplanes, will be called the **Radon transform** of the function f and denoted by the symbols $\check{f}$ or $\mathrm{P}[f]$ (for the precise definition see below). In real space the Radon transform is closely connected with the Fourier transform; namely, each of them is obtained from the other by a one-dimensional Fourier transform. However, while in the real case the advantage of the Radon transform over the Fourier transform consists mainly in the fact that, while preserving almost all the merits of the Fourier transform, it is more geometric, in the complex case the Radon transform already has an unquestionable advantage over the Fourier transform. An analogue of the Radon transform can be defined, besides affine space, also in other homogeneous spaces, where this transform is the basis of representation theory (5). From this more general point of view the advantage of the Radon transform over the Fourier transform becomes quite clear. If one regards the ordinary Fourier transform as consisting of two transforms—the Radon transform and the one-dimensional Fourier transform—then, whereas an analogue of the Radon transform exists in all homogeneous spaces, the second transform is typical only for Euclidean space. Strictly speaking, an analogue of this second transform also exists in other homogeneous spaces, but it is already connected with the theory of infinite-dimensional representations and is carried out differently for different spaces. Thus there is a definite reason to try to construct an operational calculus on the basis of the Radon transform.

In the present note an analogue of the Paley-Wiener theorem is formulated for the Radon transform in affine space, and a definition is given of the Radon transform of a generalized function. Since this is analysis, the apparatus itself is very important. Several formulas are given as examples.

2. The Radon transform of a function $f(x)$, $x = (x_1, \dots, x_n)$, in an n-dimensional **real** affine space is given by the formula

$$\check{f}(\xi; s) = \int f(x)\,\delta(s - (\xi, x))\,dx;[1] \tag{1}$$

where $\xi = (\xi_1, \dots, \xi_n)$, $(\xi, x) = \xi_1 x_1 + \dots + \xi_{nx} n$, $dx = dx_1 \cdots dx_n$, and $\delta(s)$ is the delta function. It is clear that (1) may also be written as the integral of the function $f(x)$ over the hyperplane $(\xi, x) = s$. The Radon transform of a function $f(z)$, $z = (z_1, \dots, z_n)$, in an n-dimensional **complex** affine space is given by the analogous formula

$$\check{f}(\zeta; s) = \int f(z)\, \delta(s - (\zeta, z))\, dz\, d\bar{z},$$

where $\zeta = (\zeta_1, \dots, \zeta_n)$, $(\zeta, z) = \zeta_1 z_1 + \dots + \zeta_{nz} n$, $dz = dz_1 \dots dz_n$, $\delta(s)$ is a generalized function of the complex variable s, defined by the equality $(\delta(s), \varphi(s)) = \varphi(0)$. We shall assume the functions f to be infinitely differentiable and rapidly decreasing, together with all derivatives, with respect to $x_1, \dots, x_n$ (respectively, with respect to $z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n$)*.

The function $f(z)$ is expressed in terms of its Radon transform by the following inversion formula:

$$f(z) = \frac{(-1)^{n-1}}{(2\pi)^{2n-2}} \int_{\Xi} \psi(\zeta; (\zeta, z))\, \omega_\zeta, \qquad \text{where} \quad \psi(\zeta; s) = \frac{\partial^{2n-2} \check{f}(\zeta; s)}{\partial s^{n-1} \partial \bar{s}^{n-1}}. \tag{3}$$

The integral is taken over an arbitrary surface Ξ of real dimension $2n-2$ in the space of ζ, intersecting at one point every complex line passing through the point $\zeta = 0$ (with the possible exception of a set of lines of lower dimension). For example, for Ξ one may take any hyperplane not passing through the origin. The differential form $\frac{1}{2n}\omega_\zeta$ expresses the volume of the "solid angle" with vertex at the point $\zeta = 0$, based on an element of the surface (the "solid angle" is the collection of all possible points $\lambda\zeta$, where ζ runs through an element of the surface and $|\lambda| \le 1$)**. The inversion formula for real space is well known (1–3).

Theorem 1. In order that the function $\varphi(\xi; s)$ be the Radon transform of an infinitely differentiable function in real space, rapidly decreasing together with all derivatives, it is necessary and sufficient that the following conditions be satisfied: 1) $\varphi(a\xi; as) = |a|^{-1}\varphi(\xi; s)$ for any $a \neq 0$; 2) the function $\varphi(\xi; s)$ is infinitely differentiable with respect to $\xi_1, \dots, \xi_n$ and s for $(\xi_1, \dots, \xi_n) \neq 0$; 3) for any derivative $D\varphi$ of the function φ with respect to ξ, s and any $m > 0$, as $|s| \to \infty$ we have

$$|D\varphi(\xi; s)| = o(|s|^{-m})$$

uniformly in ξ, when ξ ranges over a compact domain in the space with the point $\xi = 0$ removed; 4) the integral

$$\int_{-\infty}^{\infty} \varphi(\xi; s) s^k\, ds$$

is a homogeneous polynomial in ξ of degree k $(k = 0, 1, \ldots)$.

Theorem 2. In order that the function $\varphi(\zeta; s)$ be the Radon transform of an infinitely differentiable function in complex space, rapidly decreasing together with all derivatives, it is necessary and sufficient that the following conditions be satisfied: 1) $\varphi(a\zeta; as) = a^{-1}\bar{a}^{-1}\varphi(\zeta; s)$ for any complex $a \neq 0$; 2) the function $\varphi(\zeta; s)$ is infinitely differentiable with respect to $\zeta_1, \ldots, \zeta_n, s$ and $\bar{\zeta}_1, \ldots, \bar{\zeta}_n, \bar{s}$ for $(\zeta_1, \ldots, \zeta_n) \neq 0$; 3) for any derivative $D\varphi$ of the function φ with respect to ζ, s and $\bar{\zeta}, \bar{s}$, and for any $m > 0$, as $|s| \to \infty$ we have

$$|D\varphi(\zeta; s)| = o(|s|^{-m})$$

uniformly in ζ, when ζ ranges over a compact domain in the space with the point $\zeta = 0$ removed; 4) the integral

$$\int \varphi(\zeta; s) s^k \bar{s}^l \, ds \, d\bar{s}$$

is a homogeneous polynomial in ζ of degree k and a homogeneous polynomial in $\bar{\zeta}$ of degree l $(k, l = 0, 1, \ldots)$.

Analogous theorems hold for the Radon transforms of finite functions. One need only replace condition 3) of rapid decrease in s condition of finiteness. Theorems of this type we shall call analogues of the Paley –Wiener theorem. It is essential that, in addition to the natural conditions on decay and differentiability, the function φ satisfies additional relations (condition 4)). These relations are connected with the existence of finite-dimensional representations of the group of motions of affine space.*

3. For functions $f(z)$ in complex space and their Radon transforms there is an analogue of the Plancherel formula

$$\int |f(z)|^2 \, dz \, d\bar{z} = \frac{1}{(2\pi)^{2n-2}} \int \left| \frac{\partial^{n-1} \check{f}(\xi; s)}{\partial s^{n-1}} \right|^2 \omega_\xi \, ds \, d\bar{s}, \tag{4}$$

An analogue of the Plancherel formula for the real case was given by Yu. G. Reshetnyak. In this case the formula is written differently for spaces of even

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* A function $f(z)$ is called rapidly decreasing if, as $|z| = (|z_1|^2 + \ldots + |z_n|^2)^{1/2} \to \infty$, we have $|f(z)| = o(|z|^{-k})$ for any $k > 0$.

** The function $\varphi(\zeta) = \psi(\zeta; (\zeta, z))$ satisfies the homogeneity condition: $\varphi(a\zeta) = a^{-n}\bar{a}^{-n}\varphi(\zeta)$ for any $a \neq 0$. It follows from this that the integral $\int_{\Xi} \varphi(\zeta)\omega_\zeta$ does not depend on the choice of the surface Ξ. It is natural to call it the residue of the homogeneous function $\varphi(\zeta)$ (see (3)).

and odd dimension, and in the simplest (odd-dimensional) case it is analogous to formula (4).

4. We define the Radon transform of a generalized function in complex space so that for test functions the definition coincides with the usual one. We shall start from the formula

$$\int F(z) f(z)\, dz\, d\bar{z} = \frac{(-1)^{n-1}}{(2\pi)^{2n-2}} \int \check{F}(\xi; s)\, \check{f}_s^{(n-1,n-1)}(\xi; s)\, \omega_\xi\, ds\, d\bar{s}, \qquad (5)$$

which follows directly from (4). We write formula (5) in the form $(F, f) = (\check{F}, \check{f}_s^{(n-1,n-1)})$. By the **Radon transform of the generalized function** F we shall mean the functional $\check{F}$, given on the set of functions $\check{f}_s^{(n-1,n-1)}$, where $\check{f}$ runs through the Radon transforms of test functions, and defined by the equality $(\check{F}, \check{f}_s^{(n-1,n-1)}) = (F, f)$. For the time being the functional $\check{F}$ is defined on the subspace of "test functions" satisfying the additional relations. It can then be extended in various ways to the whole space of test functions. Thus the Radon transform of a generalized function is a generalized function in the usual sense of the word, defined nonuniquely. It is easy to show that the Radon transform of a generalized function is determined up to a summand of the form

$$\sum_{k,l=1}^{\infty} s^{k-1} \bar{s}^{l-1} a_{-k,-l}(\xi).$$

Here, when $k < n$ or $l < n$, $a_{-k,-l}(\xi)$ is an arbitrary function satisfying the homogeneity condition:

$$a_{-k,-l}(\alpha\xi) = \alpha^{-k} \bar{\alpha}^{-l} a_{-k,-l}(\xi).$$

When $k \geq n$ and $l \geq n$, the function $a_{-k,-l}(\xi)$ additionally satisfies the following condition: for every homogeneous polynomial $P_{k-n,l-n}(\xi)$ (of degree $k-n$ with respect to ξ and of degree $l-n$ with respect to $\bar{\xi}$),

$$\operatorname{Res}\,[P_{k-n,l-n}(\xi) a_{-k,-l}(\xi)] = 0.$$

The Radon transform of a generalized function in real space is defined analogously. In contrast to the complex case, here it is necessary to distinguish between spaces of even and odd dimension.

5. **Examples** (real space is assumed to be odd-dimensional).

1) $\mathbf{P}[1] = (-1)^{(n-1)/2} 2^n \pi^{n-1} \Gamma^{-1}(n)\, s^{n-1} a(\xi)$, where $a(\xi)$ is an arbitrary even homogeneous function of degree of homogeneity $-n$ with residue 1.

2) $\mathbf{P}[\delta(x_1, \dots, x_n)] = \delta(s)$.

3) $\mathbf{P}[\delta(x_1, \dots, x_k)] = \pi^{n-k-1/2} \Gamma\left(\frac{k-n+1}{2}\right) \Gamma^{-1}\left(\frac{n-k}{2}\right) |s|^{n-k-1} \delta(\xi_{k+1}, \dots, \xi_n)$

for odd k $(k < n)$;

$$\mathbf{P}\left[\delta(x_1, \dots, x_k)\right] = (-1)^{(n-k+1)/2} 2\pi^{n-k-1/2}\Gamma^{-1}\left(\frac{n-k+1}{2}\right)\Gamma^{-1}\left(\frac{n-k}{2}\right) s^{n-k-1}\ln|s|\,\delta(\xi_{k+1}, \dots, \xi_n)$$

for even k.

4)

$$\mathbf{P}\left[\theta(x_1)\right] = 2^{-1}\pi^{n-3/2}\Gamma\left(1-\frac{n}{2}\right)\Gamma^{-1}\left(\frac{n+1}{2}\right)\left[s_+^{n-1}(\xi_1)_+^{-1} + s_-^{n-1}(\xi_1)_-^{-1}\right]\delta(\xi_2, \dots, \xi_n),$$

where $\theta(x_1) = 1$ for $x_1 > 0$ and $\theta(x_1) = 0$ for $x_1 < 0$.

5)

$$\mathbf{P}[P_+^\lambda] = (-1)^{(n-1)/2}\pi^{(n-1)/2}\left[(\lambda+1)\dots\left(\lambda+\frac{n-1}{2}\right)\sqrt{|\Delta|}\,\sin\pi\lambda\right]^{-1}\times$$

$$\times|s|^{2\lambda+n-1}\left[\sin\pi\left(\frac{q}{2}+\lambda\right)Q_+^{-\lambda-n/2} - \sin\frac{\pi p}{2}\,Q_-^{-\lambda-n/2}\right],$$

where $P(x)$ is a quadratic form in a real space with discriminant $\Delta \neq 0$, having in its canonical representation p positive and q negative squares; $Q(\xi)$ is the dual form (it is assumed that λ is neither an integer nor a half-integer)*.

6)

$$\mathbf{P}[(P+c)_+^\lambda] = (-1)^{(n-1)/2}\pi^{(n-1)/2}\left[(\lambda+1)\dots\left(\lambda+\frac{n-1}{2}\right)\sqrt{|\Delta|}\,\sin\pi\lambda\right]^{-1}\times$$

$$\times\left[\sin\pi\left(\frac{q}{2}+\lambda\right)Q_+^{-\lambda-n/2}(s^2+cQ)_+^{\lambda+(n-1)/2} - \sin\frac{\pi p}{2}Q_-^{-\lambda-n/2}(s^2+cQ)_+^{\lambda+(n-1)/2}\right.$$

$$\left.+\sin\pi\left(\frac{q-1}{2}+\lambda\right)Q_-^{-\lambda-n/2}(s^2+cQ)_-^{\lambda+(n-1)/2} - \sin\frac{\pi(p-1)}{2}Q_+^{-\lambda-n/2}(s^2+cQ)_-^{\lambda+(n-1)/2}\right].$$

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* Just as, for analogues of the Radon transforms in the group of motions of Lobachevsky space, the relations are connected with the existence of degenerate representations of the group of motions [4].

7) The Radon transform of the characteristic function of the cone $x_1^2 - x_2^2 - \ldots - x_n^2 > 0$,

$$\mathbf{P}[(x_1^2 - x_2^2 - \ldots - x_n^2)_+^0] = (-1)^{(n+1)/2} 2\pi^{(n-3)/2} \Gamma^{-1}\left(\frac{n+1}{2}\right) s^{n-1} \ln|s| \, (\xi_1^2 - \xi_2^2 - \ldots - \xi_n^2)^{-n/2} + \mathbf{P}[1].$$

8)

$$\mathbf{P}[(x_1^2 - x_2^2 - \ldots - x_n^2 - 1)_+^0] = (-1)^{(n+1)/2} 2\pi^{(n-3)/2} \Gamma^{-1}\left(\frac{n+1}{2}\right) \times$$

$$\times (\xi_1^2 - \xi_2^2 - \ldots - \xi_n^2)^{-n/2} (s^2 - \xi_1^2 + \xi_2^2 + \ldots + \xi_n^2)^{(n-1)/2} \ln|s| + \mathbf{P}[1].$$

9)

$$\mathbf{P}[\delta(x_1^2 - x_2^2 - \ldots - x_n^2)] = (-1)^{(n-1)/2} 2\pi^{(n-3)/2} \Gamma^{-1}\left(\frac{n-1}{2}\right) s^{n-3} \ln|s| \, (\xi_1^2 - \xi_2^2 - \ldots - \xi_n^2)^{-n/2+1}.$$

10)

$$\mathbf{P}[\mathcal{P}^{\lambda} \overline{\mathcal{P}}^{\mu}] = c(\lambda, \mu) \, |\Delta|^{-1} s^{2\lambda+n-1} \bar{s}^{2\mu+n-1} \mathcal{Q}^{-\lambda-n/2} \overline{\mathcal{Q}}^{-\mu-n/2},$$

where $\mathcal{P}(z)$ is a quadratic form in a complex space with discriminant $\Delta \neq 0$, and $\mathcal{Q}(\zeta)$ is the dual form (it is assumed that $\lambda - \mu$ is an integer and that λ and μ are not integers).

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[*] For the definition of the generalized functions P_+^{λ}, $(P+c)_+^{\lambda}$, etc., see (3).

Note: Figure translations are in progress. See original paper for figures.