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Abstract

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ON COMPLETING GROUPS

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Using the concept of a free product with amalgamated subgroup, B. Neumann proved ([1]) that an arbitrary group can be completed, i.e., included as a subgroup in some complete group. Here and below, by a complete group we mean a group G in which the equation $x^n = g$ is solvable for every element $g \in G$ and every natural number n. Apparently, of greater interest are theorems on embedding an arbitrary group of a given class in a complete group of the same class. The most significant among theorems of this kind are A. I. Mal' cev' s theorems on completing torsion-free nilpotent and locally nilpotent groups, proved with the aid of the apparatus of the theory of groups and Lie algebras ([2]).

In the present note a new method of completing groups is proposed and, with the aid of this method, it is proved in particular that every solvable (locally solvable, periodic locally nilpotent) group is contained in some complete solvable (locally solvable, periodic locally nilpotent) group.

1. Let G be some group of transformations of a set M, and let n be an arbitrary natural number. For elements of the group G we shall use the notation

$$g\binom{\alpha}{\alpha g},$$

meaning that the element g sends the symbol α to the symbol $\alpha' = \alpha g$ $(\alpha, \alpha' \in M)$. Let, further, $M_1, M_2, \dots, M_n$ be pairwise disjoint sets, equipotent to the set M, and let φ_i be some one-to-one mapping of the set M onto M_i.

We introduce into consideration an isomorphism τ of the group G onto a subgroup $\overline{G}$ of the group S of all transformations of the set $\bigcup_i M_i$, assigning to the element

$$g = \binom{\alpha}{\alpha g}$$

of the group G the element

$$\overline{g} = \binom{\varphi_1(\alpha)}{\varphi_1(\alpha g)} \cdots \binom{\varphi_n(\alpha)}{\varphi_n(\alpha g)}$$

of the group $\overline{G} \subset S$. We shall call the isomorphism τ the diagonal n-isomorphism of the group G into the group S.

Along with the isomorphism τ, let us consider also the isomorphism τ_i. By definition, the isomorphism τ_i assigns to an arbitrary element

$$\begin{pmatrix} \alpha \\ \alpha g \end{pmatrix}$$

of the group G the element

$$\begin{pmatrix} \varphi_i(\alpha) \\ \varphi_i(\alpha g) \end{pmatrix}$$

of the group S. Everywhere below the subgroup of the group S generated by the subgroups $\tau_1(B), \tau_2(B), \ldots, \tau_n(B)$, where $B \subset G$, will be denoted by $\widetilde{B}$. Obviously, the group $\widetilde{B}$ decomposes into the direct product of the subgroups $\tau_i(B)$.

Let us mark in the group G an arbitrary element a. It is verified directly that the element

$$x = \begin{pmatrix} \varphi_1(a)\,\varphi_2(a) \cdots \varphi_n(a) \\ \varphi_2(a)\,\varphi_3(a) \cdots \varphi_1(aa) \end{pmatrix}$$

of the group S satisfies the relation $x^n = \overline{a}$, where $\overline{a} = \tau(a)$.

Lemma 1. If a subgroup N is invariant in the group G, then the subgroup $\widetilde{N}$ is invariant in the group $\langle x, \widetilde{G} \rangle$.

Lemma 2. The commutant of the group $\langle x, \widetilde{G} \rangle$ is contained in the commutant of the group $\widetilde{G}$.

Indeed, by Lemma 1 the group $\widetilde{G}$ and, consequently, its commutant $\widetilde{G}'$ are normal divisors in the group $\langle \widetilde{G}, x \rangle$. Hence, from the known commutator relations

$$[a, bc] = [a, c]c^{-1}[a, b]c,$$

$$[ab, c] = b^{-1}[a, c]b[b, c]$$

it follows that, in order to prove the lemma, it is enough to show that all commutators of the form $[\tilde{g}, x]$, $\tilde{g} \in \widetilde{G}$, belong to the commutant $\widetilde{G}'$. The latter is checked directly.

We shall call a group G **limit with respect to a certain class of groups** K if G belongs to the class K or coincides with the union of a system, completely ordered by inclusion, of subgroups belonging to this class.

If G is an arbitrary group, then by $K_0(G)$ we denote the class of groups isomorphic to the group G. Suppose that for every ordinal $\beta < \alpha$ the class $K_\beta(G)$ has already been defined. Then, as $K_\alpha(G)$, we take the class of groups that are limit with respect to the class of subgroups of all possible direct products of groups belonging to the union of the classes $K_\beta(G)$, $\beta < \alpha$.

Theorem 1. An arbitrary group G is contained in such a complete group G_1 whose commutant G_1' belongs to the class $K_\alpha(G')$ for some ordinal α.

Corollary 1. Every solvable group G is contained in some complete solvable group of the same degree of solvability as the group G.

Corollary 2. Every group G possessing a normal solvable system is contained in some complete group that also possesses a normal solvable system.

From the proof of Theorem 1 the following proposition follows.

Every group G possessing an invariant solvable system is contained in some complete group that also possesses an invariant solvable system.

Let us note that Theorem 1 contains both the theorem on the possibility of completing an arbitrary group and the theorem on the possibility of embedding an abelian group in a complete abelian group.

2. Let Σ denote a group property satisfying the following conditions: 1) every subgroup of a Σ-group is a Σ-group; 2) an extension of a Σ-group by means of a Σ-group has property Σ.

Following the terminology adopted in ([3]), we shall call an $L\Sigma$-group any group that locally has property Σ.

Lemma 3. An extension of an $L\Sigma$-group by means of a locally finite $L\Sigma$-group is an $L\Sigma$-group.

Let π be an arbitrary set of prime numbers. We shall call a group G π-complete*, if in it, for any element $g \in G$ and any natural number n such that the set $\pi(n)$ of prime divisors of n is contained in π, the equation $x^n = g$ is solvable.

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* In ([4]), by a π-complete group is meant any group G in which, for any element $g \in G$ and any natural number m, the equation $x^{p^m} = g$, $p \in \pi$, is solvable.

Theorem 2. *If the class of Σ-groups contains all possible cyclic π-groups, then every $L\Sigma$-group G is contained in some π-complete $L\Sigma$-group.*

Let the group G and, consequently, the group $\widetilde{G}$ (see the notation in § 1) be $L\Sigma$-groups, and let $\pi(n) \subset \pi$. Since, by Lemma 1, the group $\widetilde{G}$ is invariant in the group $\{x, \overline{G}\}$, the intersection

$$N = \{\overline{G}, x\} \cap \widetilde{G}$$

is invariant in $\{x, \widetilde{G}\}$, and moreover $\overline{G} \subset N$. In view of the isomorphism

$$\{x, \widetilde{G}\}/N = \{x, N\}/N \simeq \{x\}/\langle x^n \rangle$$

and the hypothesis of the theorem, the factor group $\{x, \widetilde{G}\}/N$ has the property $L\Sigma$. It follows from Lemma 3 that the group $\{x, \widetilde{G}\}$, being an extension of the $L\Sigma$-group N by means of the $L\Sigma$-group $\{x, \widetilde{G}\}/N$, will itself be an $L\Sigma$-group.

Thus, the $L\Sigma$-group G can be embedded in such an $L\Sigma$-group in which the n-th root of an arbitrarily chosen element $a \in G$ is extracted. Hence, by the usual method (see, for example, [5], p. 437), we embed the group G in a π-complete $L\Sigma$-group, as was required.

Corollary 1. Every locally solvable group is contained in some complete locally solvable group.

Corollary 2. Every (locally finite) p-group is contained in some complete (locally finite) p-group.

Since the class of periodic locally nilpotent groups coincides with the class of groups decomposable into a direct product of locally finite p-groups, from Corollary 2 we easily obtain:

Corollary 3. A periodic locally nilpotent group is contained in some complete periodic locally nilpotent group.

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Note: Figure translations are in progress. See original paper for figures.