RADICALS OF WEAKLY ASSOCIATIVE RINGS

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Abstract

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MATHEMATICS

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By a ring we shall everywhere mean a nonassociative ring. If A and B are (two-sided) ideals of a ring K, then the product AB is the smallest ideal of the ring K containing all products $a_i b_i$ ($a_i \in A,\ b_i \in B$). Powers of an ideal A are defined as follows: $A^1 = A,\ A^n = A^{n-1}A$. An ideal A is called **nilpotent** if there exists an n such that $A^n = 0$; an ideal A is called **idempotent** if $A^2 = A$.

Let K be an arbitrary ring, K_1 a subring of K, B an ideal in K_1, and A a subset of B. Then $K \supseteq K_1 \supseteq B \supseteq A$. Denote by A_B and A_{K_1} the ideals generated by the set A, respectively in the rings B and K_1. We shall call the ring K **weakly associative**, or, briefly, a **w. a. ring**, if for all A, B, and K_1 there exists a natural number n (generally depending on K_1, B, A) such that $A^n_{K_1} \subseteq A_B$. It is easy to show that associative rings are w. a. rings.

It is readily verified that the class L of all w. a. rings has the following two properties:

1) Every ideal of a ring from L is itself a ring from L.

2) Every homomorphic image of a ring from L belongs to L.

Lemma 1. Let B be an ideal in a w. a. ring K, and let A be a subset of B. If $A^2_K = A_K$, then $A_B = A_K$, i.e. A_B is an ideal in K.

Lemma 2. Let B be an ideal in a w. a. ring K, and let A be an ideal in the ring B. If the factor ring $\overline{B} = B/A$ contains no nilpotent ideals, then A will be an ideal in K.

Lemma 3. Let B be an ideal in a w. a. ring K. Then every idempotent ideal A of the ring B will be an ideal in K.

By a radical we shall mean an axiomatic radical in the sense of A. G. Kurosh and Amitsur (see [1–3]). Recall that a radical R is called **hereditary** if every ideal of an R-radical ring is an R-radical ring. A hereditary radical R is called **supernilpotent** if nilpotent rings are R-radical, i.e. if in every ring K the radical R contains all nilpotent ideals. A hereditary radical R is called **subidempotent** if R-radical rings are hereditarily idempotent, i.e. every ideal of an R-radical ring is idempotent.

Theorem 1. If $R(K)$ is a supernilpotent or subidempotent radical of a w. a. ring K, then for any ideal B of K the equality

$$R(B) = B \cap R(K)$$

holds.

Let A be an ideal in the ring B, and let B be an ideal in the ring K. Denote by $A : B$ the set of all elements x of K such that $B(x) \subseteq A$,

$(x)B \subseteq A$, where (x) is the ideal generated by the element x in K. It is clear that $A : B$ is an ideal in K.

Theorem 2. In order that a given radical R in the class of w.a. rings be supernilpotent, it is necessary and sufficient that in every w.a. ring K the equality

$$R(K) = f^{-1}[R(K/B)] \cap [R(B) : B],$$

hold, where B is an arbitrary ideal in K; f is the natural homomorphism of the ring K onto the factor ring K/B.

Recall that a nonzero ring K is called **prime** if from $AB = 0$, where A and B are ideals in K, there follows at least one of the equalities $A = 0$ or $B = 0$.

A class of rings M will be called a **special class** if the following conditions are satisfied:

I, 1. Every ring in M is a prime ring.

I, 2. Every nonzero ideal of a ring in M belongs to M.

I, 3. If at least one nonzero ideal of a prime ring K belongs to M, then K itself also belongs to M.

Let M be some class of rings. An ideal P of an arbitrary ring K will be called an M-ideal if the factor ring K/P belongs to the class M.

The class of weakly associative rings M will be a special class of rings if and only if the following conditions are satisfied:

II, 1. If an M-ideal P of a ring K does not contain an ideal A, then $(P \cap A) : A = P$.

II, 2. If an M-ideal P of a ring K does not contain an ideal A, then $P \cap A$ is an M-ideal in the ring A.

II, 3. If P_0 is an M-ideal in the ring A, where A is an ideal in K, then $P_0 : A$ is an M-ideal in K, and $P_0 = (P_0 : A) \cap A$.

If M is a special class of rings, then the corresponding M-ideals will be called **M-special ideals**.

Every special class of rings M determines an upper radical S_M in the sense of A. G. Kurosh (see ([1]) or ([4])). We shall call this radical a **special radical**. The specially radical rings are the rings that are not homomorphically mapped onto nonzero rings from the given special class of rings, i.e., rings without special ideals.

Theorem 3. In the class of weakly associative rings, every special radical S_M is a supernilpotent radical; moreover, S_M is the intersection of all M-special ideals of the ring, i.e., the S_M-semisimple rings are subdirect sums of rings from the special class M.

Theorem 4. If R is a given supernilpotent radical in the class of weakly associative rings, then the class $\widetilde{M}$ of all R-semisimple prime rings is the largest special class of rings contained in the class L of all R-semisimple rings.

Corollary. Every supernilpotent radical R in the class of w.a. rings is embedded in a special radical $S_{\widetilde{M}}$, which is the smallest among all special radicals containing R.

Let R be a given radical. We shall call a radical R' **complementary** to R if R' is the largest among all radicals having in every ring K zero intersection with R. Radicals R and S are called **mutually complementary** if $R = S'$, $S = R'$. A radical R is **self-dual** if R', R'' exist and $R = R''$.

Theorem 5. If R is a subidempotent radical in the class of w.a. rings, then R' and R'' exist, and R' is a self-dual special radical, while R'' is a self-dual subidempotent radical.

Theorem 6. If R is a supernilpotent radical in the class of w.a. rings, then R' and R'' exist, and R' is a self-dual

subidempotent radical, and R'' its dual special radical.

As usual, we shall call a ring K **subdirectly irreducible** if the intersection of all its nonzero ideals is a nonzero ideal–the heart of K. The heart of a ring K is either an idempotent or a nilpotent ideal. A class of subdirectly irreducible w.a. rings M will be special if and only if every ring in M has an idempotent heart possessing an arbitrary fixed property φ.

For radicals of weakly associative rings there is a duality theorem, generalizing the corresponding theorem for associative rings (see (4)).

Theorem 7. *Let M_φ be the class of subdirectly irreducible w.a. rings with idempotent heart possessing the given property φ, and let $M_{\tilde{\varphi}}$ be the class of all remaining subdirectly irreducible w.a. rings. Then the classes M_φ and $M_{\tilde{\varphi}}$ define upper radicals R_φ and $R_{\tilde{\varphi}}$, respectively, and R_φ and $R_{\tilde{\varphi}}$ are mutually complementary; R_φ is a self-dual special radical, while $R_{\tilde{\varphi}}$ is a self-dual subidempotent radical. In the manner described, all self-dual supernilpotent and subidempotent radicals are obtained.*

Let us note that the radicals R_{φ} and $R_{\tilde{\varphi}}$ have the following arithmetic characterization: R_{φ} is the intersection of all ideals of the ring whose factor rings are subdirectly irreducible with idempotent heart possessing the property φ; $R_{\tilde{\varphi}}$ is the intersection of all ideals of the ring whose factor rings are subdirectly irreducible with R_{φ}-radical heart.

Theorem 8. *If R is a given supernilpotent radical in the class of w.a. rings, then every R-semisimple ring K satisfying the maximal condition for ideals is a subdirect sum of a finite number of rings from the special class $\widetilde{M}$ of all R-semisimple primary rings.*

The class of all primary w.a. rings will be a special class of rings. The special radical determined by it is the Baer–McCoy radical R_m. R_m-semisimple rings are precisely rings without nilpotent ideals.

Hence, and from Theorem 8, it follows that every weakly associative ring without nilpotent ideals and satisfying the maximal condition for ideals is a finite subdirect sum of primary rings. R_m is not a self-dual radical.

As was proved above, the class of all subdirectly irreducible w.a. rings with idempotent heart will be a special class of rings. The upper radical R_1 determined by this class of rings is called the **antisimple radical** (see (6)). Antisimple rings are rings that cannot be mapped homomorphically onto subdirectly irreducible rings with idempotent heart. R_1-semisimple w.a. rings are precisely subdirect sums of subdirectly irreducible rings with idempotent heart.

As in the associative case, one can show that a weakly associative ring K is antisimple if and only if, in any of its homomorphic images $\overline{K}$, for every nonzero principal ideal $(\overline{a})$ the inequality $(\overline{a})^2 \neq (\overline{a})$ holds. The antisimple radical in the class of weakly associative rings is the least self-dual supernilpotent radical. According to Theorem 7, in the class of weakly associative rings there exists a subidempotent radical R_1' complementary to R_1. R_1'-radical rings are precisely hereditarily idempotent rings (f-regular rings of Blair (5)), i.e., rings in which every ideal is idempotent. R_1'-semisimple

i.e., rings that are direct sums of directly indecomposable rings with nilpotent core. The hereditarily idempotent radical R_1', obviously, is the largest subidempotent radical in the class of weakly associative rings.

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Note: Figure translations are in progress. See original paper for figures.