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Abstract

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MATHEMATICS

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ON THE REPRESENTABILITY OF SOLUTIONS OF OPERATOR EQUATIONS IN THE FORM OF CONTINUAL INTEGRALS*

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1. Let $\mathfrak{H}$ be a Hilbert space and let T be a self-adjoint positive-definite operator with domain D, dense in $\mathfrak{H}$. Assuming the operator T^{-1} to be bounded, introduce in $\mathfrak{H}$ a new metric by means of the scalar product $(x, y)_* = (T^{-1}x, T^{-1}y)$. The space N, obtained by completing $\mathfrak{H}$ with respect to the norm $\|x\|_* = \sqrt{(x, x)_*}$, will be called the **space of generalized elements**. Introducing in the linear set D the norm generated by the scalar product $(x, y)^* = (Tx, Ty)$, we turn it into a complete space—the space of basic elements ([1],[2]). The spaces D and N are conjugate. The values of the functional generated by a generalized element ξ on a basic element x can be obtained by means of the formula $\xi(x) = (Tx, \widetilde{T}^{-1}\xi)$, where $\widetilde{T}$ is the closure of the operator T in the space N.

If U is an operator in $\mathfrak{H}$ such that the operator TUT^{-1} is bounded, then the operator U is bounded also in the space D. The operator $\widetilde{U}^*$ conjugate to it in the space N is an extension of the operator U^*, conjugate to U in $\mathfrak{H}$. The operator $\widetilde{U}$ is defined analogously. For $U = U^*$ the relation $\widetilde{U} = \widetilde{U}^*$ holds.

Lemma 1. *Let the operator TUT be bounded in the space $\mathfrak{H}$:*

$$\|TUT\| < \infty. \tag{1}$$

Then the operators $\widetilde{U}$ and $\widetilde{U}^*$ map the space N into D, and for any generalized elements ξ and η the equality

$$\eta(\widetilde{U}\xi) = \xi(\widetilde{U}^*\eta). \tag{2}$$

is valid.

Let B be a self-adjoint operator in $\mathfrak{H}$ with a resolution of the identity E_λ and spectral function $\sigma(\lambda)$. Suppose that for every finite interval $[a, b)$ the condition

$$\left\|(E_b - E_a)T^{-1}\right\|_H < \infty, \tag{3}$$

is fulfilled, where $\|S\|_H$ is the absolute norm of the operator S $\left(\|S\|_H = \sum_{k=1}^{\infty} \|Se_k\|^2\right.$ for some orthonormal basis $\{e_k\}$ in the space $\mathfrak{H}$). Then, as G. I. Kats showed ([1]), for any element $f \in \mathfrak{H}$ there is a representation in the form of a weakly convergent integral in the space N,

$$f = \int_{-\infty}^{\infty} f_\lambda \, d\sigma(\lambda), \tag{4}$$

where f_λ is a one-parameter family of generalized eigen-elements of the operator B. Moreover, if $n(\lambda)$ is the multiplicity of the spectrum of the operator B at the point λ, then there exists a set of elements $\xi_1, \dots, \xi_k, \dots,\ k \le n(\lambda)$,

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such that for $\varphi, \psi \in D$ the equalities

$$\varphi_\lambda = \sum_{k=1}^{n(\lambda)} \xi_{k\lambda}(\varphi)\xi_{k\lambda}, \tag{5}$$

$$(\varphi, \psi) = \int_{-\infty}^{\infty} \sum_{k=1}^{n(\lambda)} \xi_{k\lambda}(\varphi)\overline{\xi_{k\lambda}(\psi)} \, d\sigma(\lambda) \tag{6}$$

hold.

2. Let A be a self-adjoint positive-definite operator in $\mathfrak{H}$. Suppose that, for some $m > 0$, the domain of definition D_{A^m} of the operator A^m is contained in D, and that for $x \in D_{A^m}$ the estimate

$$\|A^m x\| \ge M\|Tx\| \tag{7}$$

holds. Applying Lemma 1 to the operator $U(t) = e^{-At}$, we obtain that, for $\xi \in N$ and $t > 0$, the vector-function $x(t) = \widetilde{U}(t)\xi$ is strongly differentiable in $\mathfrak{H}$ and satisfies the differential equation

$$\frac{dx}{dt} = -Ax. \tag{8}$$

Let ξ_λ be a family of generalized eigen-elements of the operator B, whose spectrum, for simplicity of exposition, we shall for the time being regard as simple. The family of vector-functions $x_\lambda(t) = \widetilde{U}(t)\xi_\lambda$ will be called a **family of fundamental solutions** of equation (8), generated by the operator B.

Under certain assumptions concerning the operator C, fundamental solutions of the equation

$$\frac{dx}{dt} = -Ax + Cx \tag{9}$$

are constructed analogously.

In any case, they exist if, for some $m > 0$, the conditions

$$\|C\| < \infty, \qquad \|A^{mCA^{-m}}\| < \infty \tag{10}$$

are satisfied.

Using the results of [3], one can consider the more general case when the operator C has a fractional degree relative to A.

3. Let $M(\lambda_0, \nu)$ be the space of bounded functions $\lambda(t)$, defined for $0 \le t \le l$ and satisfying the conditions $\lambda(0) = \lambda_0,\ \lambda(l) = \nu$. If q is a partition of the interval $[0, l]$ by the points $0 < t_1 < t_2 < \dots < t_n < l$, and R is an ordered system of finite or infinite intervals $[a_i, b_i)$ $(i = 1, 2, \dots, n)$, then by $Q(q, R)$ we denote the set of functions $\lambda(t) \in M(\lambda_0, \nu)$ satisfying the conditions $a_i \le \lambda(t_i) < b_i$ $(i = 1, 2, \dots, n)$ (a cylindrical set). If conditions (3) and (7) are fulfilled, and if for any interval $[a, b)$ the set of principal elements into which the operator $E_b - E_a$ maps D is dense in D, then on cylindrical sets there is defined a function (generally speaking, complex-valued)

$$\mu_{A,B}Q(q,R) = \int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} \xi_{\lambda_1}(e^{-At_1}\xi_{\lambda_0})\xi_{\lambda_2}(e^{-A(t_2-t_1)}\xi_{\lambda_1}) \cdots$$

$$\cdots \xi_\nu(e^{-A(l-t_n)}\xi_{\lambda_n})\, d\sigma(\lambda_1) \cdots d\sigma(\lambda_n). \tag{11}$$

For any measurable bounded function $\varphi(\lambda_1, \lambda_2, \dots, \lambda_n)$, decreasing sufficiently rapidly at infinity (if $\|T^{-1}\|_H < \infty$, its boundedness is sufficient), there exists the integral

$$\int_{R_n} \varphi(\lambda_1, \dots, \lambda_n)\, \mu_{A,B}Q(q, d\lambda)$$

with respect to the complex-valued measure generated by the set function $\mu^\xi_{A,B}Q(q; R)$ for a fixed partition q in the n-dimensional space R_n.

Definition. Let $\Phi[\lambda(t)]$ be a continuous functional on the space $M(\lambda_0, \nu)$. For a partition q of the interval $[0, l]$, construct the step function $\lambda_q(t)$, defined by the equalities

$$\lambda_q(t) = \lambda(t_k) = \lambda_k \quad \text{for} \quad t_{k-1} < t \leq t_k \; (k = 1, 2, \dots, n); \quad \lambda_q(t) = \nu \quad \text{for} \quad t > t_n.$$

Set

$$\Phi[\lambda_q(t)] = \Phi_q(\lambda_1, \dots, \lambda_n), \qquad I_q(\Phi) = \int_{R_n} \Phi_q(\lambda_1, \dots, \lambda_n)\, \mu_{A,B} Q(q, d\lambda).$$

If there exists the limit $I(\Phi) = \lim_{\{q\}} I_q(\Phi)$ along the directed set formed by the (finite) partitions of the interval $[0, l]$, then we shall call it the **continual integral** of the functional $\Phi[\lambda(t)]$ and denote it conditionally by the symbol

$$I(\Phi) = \int_{M(\lambda_0, \nu)} \Phi[\lambda(t)]\, d\mu_{A,B}. \tag{12}$$

The question arises of describing the classes of functionals for which the continual integral (12) exists. In the simplest case, when for all values $t > 0$, λ, and ν the condition

$$\left(e^{-At} \xi_\lambda, \xi_\nu\right) \geq 0, \tag{13}$$

is satisfied, the set function $\mu_{A,B}$ generates a nonnegative measure on a certain Borel field of sets containing all cylindrical sets. In this case the construction given above is completely analogous to the classical construction of the continual integral with respect to Wiener measure, which exists, in any case, for bounded continuous functionals Φ. In the general case we cannot draw such a conclusion. But if $\Phi[\lambda(t)]$ depends only on the values of the function $\lambda(t)$ at some finite set of points, then the integral $I(\Phi)$ exists. This fact may be interpreted as follows. We shall call the "space of basic functions on $M(\lambda_0, \nu)$" the linear topological space of functionals Φ on the space $M(\lambda_0, \nu)$, each of which depends only on the values of $\lambda(t)$ at a finite number of points and decreases sufficiently rapidly at infinity. Convergence is then defined for sequences of functionals depending on the values of $\lambda(t)$ at one and the same set of points. Continuous functionals on this "space of basic functions" will be called "generalized functions on the space $M(\lambda_0, \nu)$." The continual integral constructed above is such a generalized function.

Below we shall indicate a class of functionals, not determined by the values of $\lambda(t)$ at a finite set of points, for which $I(\Phi)$ exists.

4. Lemma 2. Let conditions (7) and (10) be satisfied. Then, for $x \in D_{A^m}$, the estimate

$$\left\|A^m \left[e^{-(A-C)t} - e^{-At}e^{Ct}\right] x\right\| = o(t) \qquad (t \to 0)$$

holds.

Consider a partition q of the interval $[0, t]$ by the points $0 = \tau_0 < \tau_1 < \cdots < \tau_n = t$, and set $U_1(t) = e^{-At}e^{Ct}$ and $U^{(q)}(t) = U_1(\tau_n - \tau_{n-1}) \cdots U_1(\tau_2 - \tau_1) U_1(\tau_1)$. Using Lemma 2, one can obtain the following representation for the operator $U(t) = e^{-(A-C)t}$.

Theorem 1. If $\xi \in N$, then

$$\tilde{U}(t) = \lim_{\{q\}} \tilde{U}^{(q)}(t)\xi, \tag{14}$$

where the limit is understood in the sense of strong convergence of basic elements.

Suppose that in the differential equation (9) the operator C has the form

$$C = V(B) = \int_{-\infty}^{\infty} V(\lambda)\, dE_\lambda,$$

where B is an operator satisfying condition (3), and $V(\lambda)$ is some function.

Theorem 2. The fundamental solution $x_a(t) = e^{-(A-C)t}\xi_a$ of the differential equation

$$\frac{dx}{dt} = -Ax + V(B)x, \tag{9'}$$

whose coefficients satisfy conditions (3), (7), and (10), is representable in the form of the continual integral

$$x_a(t) = \int_{-\infty}^{\infty} \left[\int_{M(a,\nu)} \exp\left[\int_0^t V(\lambda(\tau))\, d\tau\right] d\mu_{A,B}\right] \xi_\nu \, d\sigma(\nu).$$

5. Under the previous assumptions concerning the operators, consider the differential equation

$$\frac{dx}{dt} = [iA + V(B)]x. \tag{15}$$

We shall use the device indicated by Feynman [4] (see also [5]) and introduce the equation

$$\frac{dx}{dt} = (i - \varepsilon)Ax + Cx. \qquad (16)$$

For it one can carry out all the arguments set forth above. Since the fundamental solutions of this equation tend, in the sense of strong convergence in N, to the corresponding fundamental solutions $x_a(t)$ of equation (15), the representation is valid

$$x_a(t) = \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\left[\int_{M(a,\nu)}\exp\left[\int_0^t V(\lambda(\tau))\,d\tau\right]d\mu_{A_\varepsilon,B}\right]\xi_\nu\,d\sigma(\nu), \qquad A_\varepsilon = (i-\varepsilon)A,$$

converging in the same sense.

6. Using the results of T. Kato and M. A. Krasnosel' skii, S. G. Krein, and P. E. Sobolevskii [6–8], one can transfer the results obtained to equations with coefficients depending on t.

Because of lack of space, we do not set out here applications of the results considered to differential equations and systems of parabolic type.

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Note: Figure translations are in progress. See original paper for figures.