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Abstract

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ON THE SECOND OBSTRUCTION FOR SECANT SURFACES

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The present note contains a proof of a formula for the second obstruction to extending a secant surface of a skew product. The formula obtained by us generalizes a formula of Liao, established by him only for the case of bundles of spheres ([3]).

1. Formulas of V. G. Boltyansky and S. D. Liao

Let $\mathcal{E}\{E, B, F, p, G\}$ be a skew product with base B, fiber F, group G, and projection $p : E \to B$. In what follows we shall assume that the fiber F is aspherical in dimensions $< n$ (i.e., that it is linearly connected and $\pi_i(F) = 0$ for $i < n$) and simple in dimensions $\leq n+1$. As is known, in this case the first obstruction $Z_{\mathcal{E}}^{n+1} \in H^{n+1}(B; \pi_n(F))$ to extending a secant surface is defined. If $Z_{\mathcal{E}}^{n+1} = 0$, then a secant surface can be constructed over an $(n+1)$-dimensional skeleton B^{n+1} of the base B. In this case there arises a second obstruction to extending the secant surface, and it will already depend not only on the skew product itself, but also on the choice of the secant surface over B^n. Under certain assumptions concerning F and G, Boltyansky established a formula ([1]) which makes it possible to survey the set of all second obstructions. Namely:

$$
\begin{aligned}
Z_{\mathcal{E}}^{n+2}(f) &= \operatorname{Sq}^2 D^n(f_0, f) + D^n(f_0, f) \smile X^2 + Z_{\mathcal{E}}^{n+2}(f_0), \quad n > 2; \\
Z_{\mathcal{E}}^{n+2}(f) &= K_B^2 D^n(f_0, f) + D^n(f_0, f) \smile X_f^2 + Z_{\mathcal{E}}^{n+2}(f_0), \quad n = 2,
\end{aligned} \tag{1}
$$

where the multiplication operations and $\operatorname{Sq}^2, K_B^2$ are defined by means of certain pairs of groups ([2]); $D^n(f_0, f)$ is the cohomology class distinguishing the secant surfaces f_0 and f, and the two-dimensional cohomology class ([1]) X^2 for $n = 2$ also depends on f_0.

In the case where $\mathcal{E}$ is a bundle of n-dimensional spheres, Liao proved the formula

$$
\begin{aligned}
p^* Z_{\mathcal{E}}^{n+2}(f) &= \operatorname{Sq}^2 \theta(f) + \theta(f) \smile \beta \operatorname{Sq}^2 \theta(f), \quad n > 2; \\
p^* Z_{\mathcal{E}}^{n+2}(f) &= \theta(f) \smile \theta(f) + \theta(f) \smile \beta[\theta(f) \smile \theta(f)], \quad n = 2.
\end{aligned} \tag{2}
$$

We shall give the definition of the element $\theta(f)$ below. Concerning the homomorphism $\beta : H^{n+2}(E, \pi_{m+1}(F)) \to H^2(E, \pi_m(F))$, see ([3]).

In the present note we shall show, using formulas (1), that formulas (2) also hold in the general case, and not only when the fiber is a sphere.

2. Definition of the class $\theta(f)$

Let $k_b^n \in H^n(p^{-1}(b), \pi_n(F))$ be the fundamental class of the fiber $p^{-1}(b)$, $b \in B$. Consider the class $k^n \in H^0(B, \pi_n(F))$, defined by the relation $k^n(b) = k_b^n$, $b \in B$. The class k^n may be assumed to belong to the term $E_2^{0,n}$ of the spectral sequence of the -

fibration $\mathfrak{G}$; moreover, it is known that the transgression $d_{n+1} : E_2^{0,n} \to E_2^{n+1,0}$ carries it to the first obstruction, i.e. $d_{n+1}(k^n) = Z_{\mathfrak{G}}^{n+1}$. In what follows we shall assume that $Z_{\mathfrak{G}}^{n+1} = 0$, whence it follows that $d_{n+1}(k^n) = 0$ and the transgression d_{n+1} carries the whole group $E_2^{0,n} \simeq H^n(F, \pi_n(F))$ to zero. (This is clear, for example, from the fact that every class $y \in H^n(F, \pi_n(F))$ has the form $x^0(k_F^n)$, where x^0 is a cohomology operation of degree 0, k_F^n is the fundamental class of the fiber F, and transgression commutes with operations.)

Considering the spectral sequence of the fibration $\mathfrak{G}$, we can construct the exact Serre sequence ([5], p. 21). This sequence has the form

$$0 \to E_2^{n,0} \xrightarrow{p^*} H^n(E, \pi_n(F)) \xrightarrow{\beta} E_2^{0,n} \xrightarrow{d_{n+1}} E_2^{n+1,0}.$$

Replacing in this sequence the groups $E_2^{p,q}$ by groups isomorphic to them, we obtain the following exact sequence:

$$0 \to H^n(B, \pi_n(F)) \xrightarrow{p^*} H^n(E, \pi_n(F)) \xrightarrow{\beta} H^0(B; H^n(F, \pi_n(F))) \xrightarrow{d_{n+1}} H^{n+1}(B, \pi_n(F)). \qquad (\Sigma)$$

The homomorphism β is defined by the relation $(\beta y)(b) = i_b^* y$, where $i_b : p^{-1}(b) \to E$ is the inclusion map $(b \in B)$.

By the exactness of the sequence (Σ) and the equality $d_{n+1}(k^n) = 0$, we find that $k^n \in \operatorname{Im} \beta$. Let $a \in \beta^{-1}(k^n)$; then set

$$\theta(f) = a - p^* f^* a.$$

This definition is correct. Indeed, if $b \in \beta^{-1}(k^n)$, then $b - p^* f^* b = a - p^* f^* a + (b - a) - p^* f^* (b - a)$, where the element $b - a$, by the exactness of the sequence (Σ), has the form $p^* c$, and therefore $(b - a) - p^* f^* (b - a) = p^* c - p^* f^* p^* c = 0$, since $f^* p^* = 1$.

Further, $f^*\theta(f) = f^*[a - p^*f^*a] = f^*a - f^*p^*f^*a = 0$. Thus, by the definition of the element $\theta(f)$, we obtain the following lemma:

Lemma 1. The class $\theta(f)$ is uniquely determined by the relations

$$\beta\theta(f) = k^n, \qquad f^*\theta(f) = 0.$$

3. Main lemma. Consider the fiber product $\widetilde{\mathfrak{G}} = \{\widetilde{E}, E, F, \tilde{p}, G\}$, induced by the projection p ([4]). Recall that the points of the space $\widetilde{E}$ are pairs (e_1, e_2) satisfying the condition $p(e_1) = p(e_2)$. The projection $\tilde{p} : \widetilde{E} \to E$ is defined by the equality $\tilde{p}(e_1, e_2) = e_1$. The product $\mathfrak{G}$ has a canonical cross-section $\tilde{f}_0$, defined by the equality $\tilde{f}_0(e) = (e, e)$. Since this cross-section is defined over the entire base E of the fiber product $\widetilde{\mathfrak{G}}$, we have

$$Z^{n+2}_{\widetilde{\mathfrak{G}}}(\tilde{f}_0) = 0. \tag{3}$$

Further, every cross-section f of the product $\mathfrak{G}$, given over B^{n+1}, defines an induced cross-section $\tilde{f}$ of the product $\widetilde{\mathfrak{G}}$, defined over $p^{-1}(B^{n+1})$ by the equality $\tilde{f}(e) = (e, fp(e))$. We may suppose that $E^{n+1} \subset p^{-1}(B^{n+1})$. Indeed, this will be the case when p is a simplicial map. But this can always be achieved by using singular theory. Hence one may speak of the second obstruction $Z^{n+2}_{\widetilde{\mathfrak{G}}}(\tilde{f})$. From the general properties of ob

follows from the obstruction

$$Z^{n+2}_{E}(\tilde{f}) = p^* Z^{n+2}_{E}(f). \tag{4}$$

We now prove the validity of the following lemma.

Lemma 2 (main). $\theta(f) = D^n(\tilde{f}_0, \tilde{f})$.

For the proof, according to Lemma 1, it suffices to establish that

$$f^* D^n(\tilde{f}_0, \tilde{f}) = 0, \qquad \beta D^n(\tilde{f}_0, \tilde{f}) = k^n.$$

To prove the relation $f^*D^n(\tilde{f}_0, \tilde{f}) = 0$, it is enough to show that over $f(B^n)$ the secant surfaces $\tilde{f}_0$ and $\tilde{f}$ coincide. Putting $e = f(b)$, $b \in B$, we obtain

$$\tilde{f}(e) = (e, fp(e)) = (e, f(b)) = (e, e) = \tilde{f}_0(e).$$

To prove the second relation it is necessary to check, by the definition of the homomorphism β, that $i_b^* D^n(f_0, \tilde{f}) = k_b^n$. Considering the part of the product $\tilde{\mathcal{E}}$ over $p^{-1}(b)$, we see that this is equivalent to the direct product $p^{-1}(b) \times p^{-1}(b)$

($\tilde{p}$ is the projection onto the first summand). The secant surface $\tilde{f}_0$, considered over $p^{-1}(b)$, is the diagonal in the product $p^{-1}(b)\times p^{-1}(b)$, while $\tilde{f}$ is the graph of the constant map $p^{-1}(b)\to f(b)$. Since the diagonal is the graph of the identity map, the difference $D^n(f_0,\tilde{f})$, considered only over $p^{-1}(b)$, is the difference between the constant and the identity maps of the layer $p^{-1}(b)$ into itself, i.e. the fundamental class k_b^n. The lemma is proved.

4. Proof of the main theorem

Theorem

$$p^*Z_E^{n+2}(f)=\operatorname{Sq}^2\theta(f)+\theta(f)\smile p^*X^2,\qquad n>2;$$

$$p^*Z_E^{n+2}(f)=K_B^2\theta(f)+\theta(f)\smile p^*X_f^2,\qquad n=2. \tag{5}$$

For the proof we apply formula (1) to the secant surfaces $\tilde{f}_0$ and $\tilde{f}$. We obtain, by virtue of (3):

$$Z_{\tilde{\mathcal{E}}}^{n+2}(\tilde{f})=\begin{cases}\operatorname{Sq}^2 D^n(\tilde{f}_0,\tilde{f})+D^n(\tilde{f}_0,\tilde{f})\smile\tilde{X}^2, & n>2;\\ K_B^2D^n(\tilde{f}_0,\tilde{f})+D^n(\tilde{f}_0,\tilde{f})\smile\tilde{X}_f^2, & n=2.\end{cases}$$

Hence, by the main lemma and (4), we obtain the desired result.

Remark. Our arguments carry over completely to the case of an arbitrary fibration satisfying formula (1). Fiber spaces in the sense of Serre are fibrations of this type*.

In conclusion, I express my gratitude to V. G. Boltyanskii and A. S. Schwarz for their attention to this work.

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* An unpublished result of V. G. Boltyanskii.

Note: Figure translations are in progress. See original paper for figures.

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