THE SCHWARZ ALGORITHM IN THE ELASTICITY-THEORY PROBLEM ON STRESSES

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Fig. 1

Figure 1: Fig. 1

Abstract

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THEORY OF ELASTICITY

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THE SCHWARZ ALGORITHM IN THE ELASTICITY-THEORY PROBLEM ON STRESSES

(Presented by Academician S. L. Sobolev on 17 VI 1960)

The convergence of the Schwarz algorithm in the elasticity-theory problem on displacements was proved by S. L. Sobolev ([1]).

Let us consider the application of the Schwarz algorithm to the problem on stresses in the domain D'_{12}, which is the sum of the domains D_1 and D_2, partially overlapping one another (Fig. 1). To prove the existence of the limit of the function obtained by means of the Schwarz algorithm, we shall use the method of S. L. Sobolev ([1]). The coincidence of this limit with the solution of the equations of the theory of elasticity for the domain D'_{12} will be shown on the basis of the known uniqueness theorem. In the problem on stresses, the arbitrariness in the choice of the initial vector function of stresses p_0 on the surfaces of the domain D_2 inside the domain D_1 is restricted by the condition of static equivalence of the initial stresses p_0 to the actual internal stresses on the same surfaces inside the domain D_{12}. For the case of multiply connected domains D'_{12}, this condition can be fulfilled on the basis of the application of the principle of least work.

Fig. 1

As in ([1]), we adopt the notation: S_1 is the boundary of D_1; S_2 is the boundary of D_2; S'_1 is the part of S_1 inside D_2; S''_1 is the part of S_1 outside D_2; S'_2 is the part of S_2 inside D_1; S''_2 is the part of S_2 outside D_1; D_{12} is the domain bounded by the surfaces S'_1 and S'_2; D'_1 is the part of D_1 outside D_2; D'_2 is the part of D_2 outside D_1.

We shall seek the solution of the system of equations of the theory of elasticity in stresses, which includes 3 equilibrium equations:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0, ... \tag{1}$$

and 6 equations expressing the compatibility conditions:

$$\Delta\Delta\sigma_x = 0, \qquad \Delta\Delta\sigma_y = 0, \ldots \tag{2}$$

for the simply connected domain D'_{12}, under the boundary conditions

$$p_\nu\big|_{S''_1} = q_1; \qquad p_\nu\big|_{S''_2} = q_2; \tag{3}$$

p_ν is the stress vector over surface elements; q_1, q_2 are prescribed surface forces.

The posed problem is equivalent to the problem of the calculus of variations of finding the minimum of the integral expressing the strain energy:

$$E_1(\sigma) = \frac{1}{6\mu}\int_{D'_{12}} \left\{ \frac{\mu}{3(\lambda + {}^2/{}_3\mu)}(\sigma_x + \sigma_y + \sigma_z)^2 + \tau_{yz}^2 + \tau_{xz}^2 + \tau_{xy}^2 \right.$$
$$\left. + \frac{1}{2}\left[(\sigma_z - \sigma_y)^2 + (\sigma_x - \sigma_y)^2 + (\sigma_x - \sigma_z)^2\right] \right\} dx\, dy\, dz \tag{4}$$

under the same boundary conditions (3). By σ without subscripts below we shall denote here and below the stress tensor.

Solving successively the problem separately for the domains D_2 and D_1, we obtain a sequence of functions $\sigma_x^{(2k)}$, ..., satisfying equations (1) and (2) in D_2 and D'_1 separately, and functions $\sigma_x^{(2k+1)}$, ..., satisfying equations (1) and (2) in D_1 and D'_2 likewise separately. In accordance with the idea of the Schwarz algorithm, these sequences of functions are constructed under the following boundary conditions:

$$p_\nu^{(2k)}\big|_{S''_1} = q_1; \qquad p_\nu^{(2k)}\big|_{S''_2} = q_2; \qquad p_\nu^{(2k)}\big|_{S'_1} = \begin{cases} p_0, & k = 0, \\ p_\nu^{(2k-1)}\big|_{S'_2}, & k > 0; \end{cases} \tag{5}$$

$$p_\nu^{(2k+1)}\big|_{S''_1} = q_1; \qquad p_\nu^{(2k+1)}\big|_{S''_2} = q_2; \qquad p_\nu^{(2k+1)}\big|_{S'_1} = p_\nu^{(2k)}\big|_{S'_1}, \tag{6}$$

where $p_\nu^{(2k)}, p_\nu^{(2k+1)}$ are the stress vectors on the corresponding surfaces; $k = 0, 1, 2, \ldots$ is the number of the solution for the domains D_2 and D'_1; p_0 is an arbitrary initial vector function, which must satisfy the equilibrium conditions:

$$\int_{S'_2} p_0\, ds + \int_{S''_2} q_2\, ds = 0; \qquad \int_{S'_2} [p_0, r]\, ds + \int_{S''_2} [q_2, r]\, ds = 0. \tag{7}$$

The functions $\sigma_x^{(2k)}, \ldots, \sigma_x^{(2k+1)}$, ..., constructed according to conditions (5), (6), are discontinuous in D'_{12} and, evidently, satisfy the equilibrium equations (1) in the whole domain D'_{12}. However, these functions, in general, do not satisfy equations (2) in the domain D'_{12}. The latter means that the displacements corresponding to the functions $\sigma_x^{(2k)}, \ldots$ will have a discontinuity jump on S'_2, while the displacements corresponding to $\sigma_x^{(2k+1)}, \ldots$ will acquire a discontinuity jump on S'_1.

It follows from conditions (5), (6) that $\sigma_x^{(2k)} = \sigma_x^{(2k+1)}, \ldots$ in D'_2, and moreover in D_1 the functions $\sigma_x^{(2k)}, \ldots$ do not satisfy equations (2), whereas $\sigma_x^{(2k+1)}, \ldots$ do satisfy them. Hence the inequalities follow

$$E_{D_1}(\sigma^{(2k)}) \geq E_{D_1}(\sigma^{(2k+1)}), \qquad E_{D'_{12}}(\sigma^{(2k)}) \geq E_{D'_{12}}(\sigma^{(2k+1)}). \tag{8}$$

In the same way we obtain

$$E_{D'_{12}}(\sigma^{(2k-1)}) \geq E_{D'_{12}}(\sigma^{(2k)}). \tag{9}$$

Noting that the sequence of decreasing positive numbers $E_{D'_{12}}(\sigma^{(n)})$ is convergent, it is not difficult to show in the same way as in (1) that the deformation energy corresponding to the differences $\sigma_x^{(2k)} - \sigma_x^{(2k+1)}, \ldots$, i.e. $E_{D'_{12}}(\sigma^{(2k)} - \sigma^{(2k-1)})$, and consequently the differences themselves $\sigma_x^{(2k)} - \sigma_x^{(2k-1)}, \ldots$, tend to zero in the whole domain D'_{12} as $k \to \infty$. Thus it turns out that the functions $\sigma^{(2k)}$ and $\sigma^{(2k-1)}$ have one and the same limit σ_c:

$$\lim_{k\to\infty} \sigma^{(2k)} = \sigma'_c; \qquad \lim_{k\to\infty} \sigma^{(2k-1)} = \sigma''_c; \qquad \sigma'_c = \sigma''_c = \sigma_c. \tag{10}$$

We shall show that this limit is unique and coincides with the solution of the system of equations (1) and (2) for the domain D'_{12} under the boundary conditions (3). Consider the entire set of functions which satisfy the equilibrium equations (1) in the whole domain D'_{12} under the boundary conditions (3) on S''_1 and S''_2, and satisfy equations (2) only separately in the domains D_1 and D'_2 or D_2 and D'_1, and correspond to all conceivable values of the functions on S'_1 and, respectively, S'_2.

According to the uniqueness theorem ([2]), this set consists of a subset of functions that do not satisfy equations (2) for the whole domain D'_{12}, and of one function that satisfies equations (2) in the domain D'_{12} under conditions (3).

Suppose that the limiting functions σ'_c and σ''_c are among the first subset. Then the displacements corresponding to σ'_c and σ''_c will have a discontinuity in the domain D'_{12}, with σ'_c giving a discontinuity on S'_2, and σ''_c on S'_1. Noncoincidence of the places of discontinuity means that

$$\sigma'_c \neq \sigma''_c. \tag{11}$$

Fig. 2

Figure 2: Fig. 2

Fig. 2

This result contradicts condition (10) and proves that the limiting functions $\sigma'_c = \sigma''_c = \sigma_c$ cannot be among the subset of functions that do not satisfy equations (2) in the domain D'_{12}. Hence we arrive at the conclusion that the limiting function σ_c coincides with the solution of equations (1) and (2) under conditions (3).

Now take the multiply connected domain D'_{12}. Let the simply connected domains D_1 and D_2, whose sum gives D_{12}, overlap in such a way that we have m separate domains $D_{12}^{(i)}$ (Fig. 2). We shall construct the sequence of functions $\sigma_x^{(2k)}, \ldots$ and $\sigma_x^{(2k+1)}, \ldots$ under conditions (5) and (6) in the same way as for a simply connected domain. The initial arbitrary vector functions of forces p_{0i} on the surfaces S'_{2i} must obey the equilibrium conditions:

$$\sum_{i=1}^{m} \int_{S'_{2i}} p_{0i}\, ds + \int_{S''_2} q_2\, ds = 0; \qquad \sum_{i=1}^{m} \int_{S'_{2i}} [p_{0i}, r]\, ds + \int_{S''_2} [q_2, r]\, ds = 0; \quad (12)$$

here S'_{2i} is the surface of the domain D_2 inside D_1 on the i-th overlap segment.

Take the equilibrium conditions of the domains $D_{12}^{(i)}$:

$$\int_{S'_{2i}} p_{\nu i}^{(2k-1)}\, ds + \int_{S'_{1i}} p_{\nu i}^{(2k)}\, ds = 0; \qquad \int_{S'_{2i}} [p_{\nu i}^{(2k-1)}, r]\, ds + \int_{S'_{1i}} [p_{\nu i}^{(2k)}, r]\, ds = 0, \quad (13)$$

$$\int_{S'_{2i}} p_{\nu i}^{(2k)}\, ds + \int_{S'_{1i}} p_{\nu i}^{(2k+1)}\, ds = 0; \qquad \int_{S'_{2i}} [p_{\nu i}^{(2k)}, r]\, ds + \int_{S'_{1i}} [p_{\nu i}^{(2k+1)}, r]\, ds = 0,$$

where S'_{1i} is the surface of the domain D_1 inside D_2 on the i-th overlap segment, and $p_{\nu i}^{(2k)}$, $p_{\nu i}^{(2k+1)}$ are the force vectors on the surfaces, respectively, S'_{2i} and S'_{1i}, corresponding to the functions $\sigma_x^{(2k)}, \ldots$ and $\sigma_x^{(2k+1)}, \ldots$.

It follows from (13) that on S'_{2i} all $p_{\nu i}^{(2k)}$ and their limit remain statically equivalent to the arbitrary forces $p_{0i} = p_{\nu i}^{(2k)}\big|_{k=0}$.

Thus, in order that the limiting function σ_c coincide with the solution of equations (1) and (2) for the domain D'_{12} under conditions (3), the initial forces p_{0i}

must be statically equivalent to the actual internal forces on each surface S'_{2i}. This can be accomplished with an accuracy up to 6 undetermined parameters X_r for each S'_{2i}. Taking the forces p_{0i},

for example, in the form of 3 nonparallel concentrated forces and 3 couples, and, arbitrarily choosing the points of their application, we may take as the parameters X_r the magnitudes of these forces and couples. For m overlapping domains $D_{12}^{(i)}$, the total number of unknown parameters will be $6(m-1)$. The forces on one of the surfaces S'_{2i} are expressed in terms of the parameters on the remaining $m-1$ surfaces by equations (12).

To determine the unknown parameters one may use equations following from the principle of least work:

$$\frac{\partial E_{D'_{12}}(\sigma_c)}{\partial X_r} = 0, \qquad r = 1, 2, \dots, m-1, \tag{14}$$

where $E_{D'_{12}}(\sigma_c)$ is the work of deformation, determined by formula (4) for the limiting functions $\sigma_{xc}, \sigma_{yc}, \dots$ (the components of the tensor σ_c), expressed in terms of the parameters X_r. The equations turn out to be linear with respect to X_r:

$$\sum_{r=1}^{n} \delta_{sr} X_r + \delta_{sq} = 0 \qquad (s = 1, 2, \dots, n = m-1), \tag{15}$$

where the coefficients δ_{sr} may be represented in the form

$$\delta_{sr} = \frac{1}{E} \int_{D'_{12}} [\sigma_{xr}\sigma_{xs} + \sigma_{yr}\sigma_{ys} + \sigma_{zr}\sigma_{zs} - \mu(\sigma_{xr}\sigma_{ys} + \sigma_{yr}\sigma_{xs} + \sigma_{xr}\sigma_{zs} + \sigma_{zr}\sigma_{xs} + \sigma_{yr}\sigma_{zs} + \sigma_{zr}\sigma_{ys})$$

$$+2(1+\mu)(\tau_{xyr}\tau_{xys} + \tau_{xzr}\tau_{xzs} + \tau_{yzr}\tau_{yzs})]\, dx\, dy\, dz, \tag{16}$$

where E is the modulus of elasticity and μ is Poisson' s ratio. The formula for δ_{sq} is obtained by replacing all indices r in formula (16) by the index q. The quantities $\sigma_{xr}, \sigma_{yr}, \dots$ are the limiting functions obtained by means of the Schwarz algorithm if the following parameter values are taken:

$$X_1 = X_2 = \dots = X_{r-1} = 0, \qquad X_r = 1,$$
$$X_{r+1} = \dots = X_n = 0, \qquad q_1 = q_2 = 0.$$

The functions $\sigma_{xs}, \sigma_{ys}, \dots$ correspond to the parameter values:

$$X_1 = \dots = X_{s-1} = 0, \qquad X_s = 1, \qquad X_{s+1} = \dots = X_n = 0, \qquad q_1 = q_2 = 0.$$

The functions $\sigma_{xq}, \sigma_{yq}, \ldots$ are obtained analogously with all $X_r = 0$, but with $q_1 \neq 0$ and $q_2 \neq 0$.

The well-known equations of the force method for statically indeterminate bar systems are obtained as a special case of equations (15), and the general displacement formulas follow directly from (16).

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CITED LITERATURE

1. S. L. Sobolev, DAN, 4 (13), No. 6, 235 (1936).

2. E. Trefftz, *Mathematical Theory of Elasticity*, 1934, pp. 52–56.

Note: Figure translations are in progress. See original paper for figures.