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Abstract

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ON A NUMERICAL METHOD FOR SOLVING BOUNDARY-VALUE PROBLEMS FOR PARTIAL DIFFERENTIAL EQUATIONS

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In the approximate solution of boundary-value problems for elliptic differential equations by reducing them to the corresponding finite-difference problems, one has, as a rule, to deal with a very large number of linear algebraic equations ([1–4]). In the present work special formulas are established for certain operators in partial finite differences, which make it possible to write the solutions of the corresponding systems of algebraic equations in a rather simple form, or to reduce them to systems with a small number of unknowns.

We shall use a rectangular uniform grid $x_i = x_0 + ih$, $y_k = y_0 + kh_1$ $(i, k = 0, \pm 1, \pm 2, \ldots)$, where h is the grid step in x, and h_1 is the grid step in y. Putting $h/h_1 = \alpha$, the harmonic $\Delta_h u$ and biharmonic $\Delta\Delta_h u$ finite-difference operators can be written, respectively, in the form of stencils (see, for example, ([1]))

$$\Delta_h u = \frac{1}{h^2} \begin{array}{|c|c|c|} \hline & \alpha^2 & \\ \hline 1 & -2(1+\alpha^2) & 1 \\ \hline & \alpha^2 & \\ \hline \end{array} u;$$

$$\Delta\Delta_h u = \frac{1}{h^4} \begin{array}{|c|c|c|c|c|} \hline & & \alpha^4 & & \\ \hline & 2\alpha^2 & -4(1+\alpha^2)\alpha^2 & 2\alpha^2 & \\ \hline 1 & -4(1+\alpha^2) & 6(1+\alpha^2)+8\alpha^2 & -4(1+\alpha^2) & 1 \\ \hline & 2\alpha^2 & -4(1+\alpha^2)\alpha^2 & 2\alpha^2 & \\ \hline & & \alpha^4 & & \\ \hline \end{array} u.$$

Let D be the rectangle determined by the set of points (x_i, y_k) $(i = 0, 1, \ldots, m+1;\ k = 0, 1, \ldots, n+1)$.

Introduce the notation $u_k(x) = u(x, y_k)$, $f_k(x) = f(x, y_k)$, $\lambda_k = \cos k\beta$, $a_k = 1 + \alpha^2 - \alpha^2\lambda_k$, $\beta = \dfrac{\pi}{n+1}$ $(k = 1, 2, \ldots, n)$; the n-dimensional vectors $\mathbf{u}(x) = \{u_k(x)\}$, $\mathbf{f}(x) = \{f_k(x)\}$, and the matrix of order n

$$S = \sqrt{\frac{2}{n+1}}\left(((-1)^{i+k}\sin ik\beta)\right)_1^n.$$

Theorem 1. For the general solution of the finite-difference equation

$$\Delta_h u - 2\lambda u = f(x,y) \qquad (\lambda^2 = \text{const} > 0) \tag{1}$$

in the rectangle D, the equality holds

$$\mathbf{u}(x_i) = S\mathbf{A}(x_i) + S\mathbf{B}(x_i) + \sum_{p=1}^{i-1} ST(i-p)S\left[h^2 f(x_p) - \alpha^2 \vec{\omega}(x_p)\right]$$

$$(i = 0, 1, \dots, m+1), \tag{2}$$

where the last sum for $i = 0, 1$ is equal to zero; $\vec{\omega}(x)$, $\mathbf{A}(x)$, $\mathbf{B}(x)$ are n-dimensional vectors;

$$\vec{\omega}(x) = \{u_0(x), 0, \dots, 0, u_{n+1}(x)\},$$

$$\mathbf{A}(x) = \{A_k \varphi_k(x)\}, \qquad \mathbf{B}(x) = \{B_k \psi_k(x)\};$$

A_k, B_k $(k = 1, 2, \dots, n)$ are arbitrary real constants; $\varphi_k(x)$, $\psi_k(x)$, and the diagonal matrix of order n, $T(i) = ((T_k(i)))_1^n$, depending on the quantity $\eta_k = a_k + \lambda h^2$, are determined by Table 1, where

$$\mu_k = \eta_k + \sqrt{\eta_k^2 - 1}, \qquad \nu_k = \eta_k - \sqrt{\eta_k^2 - 1}, \qquad \theta_k = \arccos \eta_k.$$

Table 1

	$\varphi_k(x_i)$	$\psi_k(x_i)$	$T_k(i)$
$\lvert\eta_k\rvert > 1$	μ_k^i	ν_k^i	$(\mu_k^i - \nu_k^i)(\mu_k - \nu_k)^{-1}$
$\lvert\eta_k\rvert = 1$	μ_k^i	$i\mu_k^i$	$i\mu_k^{i-1}$
$\lvert\eta_k\rvert < 1$	$\cos i\theta_k$	$\sin i\theta_k$	$\sin i\theta_k (\sin\theta_k)^{-1}$

Theorem 2. For the general solution of the finite-difference biharmonic equation

$$\Delta\Delta_h u = f(x,y) \tag{3}$$

in the rectangle D, the equality holds

$$\mathbf{u}(x_i) = S\mathbf{A}(x_i) + S\mathbf{B}(x_i) + S\mathbf{C}(x_i) + S\mathbf{D}(x_i) +$$

$$+\sum_{p=2}^{i-2} ST(i-p)S\left[h^2 f(x_p) - \vec{\omega}_0(x_p) - \alpha^4 \vec{\omega}_{-1}(x_p)\right] \quad (i = 0, 1, \dots, m+1), \quad (4)$$

where the last sum for $i = 0, 1, 2, 3$ is regarded as equal to zero; $\vec{\omega}_0(x)$, $\vec{\omega}_{-1}(x)$, $\mathbf{A}(x)$, $\mathbf{B}(x)$, $\mathbf{C}(x)$, $\mathbf{D}(x)$ are n-dimensional vectors:

$$\vec{\omega}_0(x) = \{Qu_0(x-2h), \alpha^4 u_0(x), 0, \dots, 0, \alpha^4 u_{n+1}(x), Qu_{n+1}(x-2h)\},$$

$$\vec{\omega}_{-1}(x) = \{u_{-1}(x) + \chi[u_{-1}(x)], 0, \dots, 0, u_{n+2}(x) + \chi^*[u_{n+2}(x)]\},$$

$$\mathbf{A}(x_i) = \{A_k \mu_k^i\}, \qquad \mathbf{B}(x_i) = \{B_k \nu_k^i\}, \qquad \mathbf{C}(x_i) = \{C_k i \mu_k^i\},$$

$$\mathbf{D}(x_i) = \{D_k i \nu_k^i\};$$

$$\mu_k = a_k + \sqrt{a_k^2 - 1}, \qquad \nu_k = a_k - \sqrt{a_k^2 - 1};$$

A_k, B_k, C_k, D_k $(k = 1, 2, \dots, n)$ are arbitrary real constants;

$$Qu_k(x-2h) = 2\alpha^2 u_k(x+h) - 4(1+\alpha^2)\alpha^2 u_k(x) + 2\alpha^2 u_k(x-h),$$

χ and χ^* are given functions of their arguments; $T(i) = \{T_k(i)\}_1^n$ is a diagonal matrix of order n,

$$T_k(i) = \left[(i-1)(\mu_k^{i+1} - \nu_k^{i+1}) - (i+1)(\mu_k^{i-1} - \nu_k^{i-1})\right](\mu_k - \nu_k)^{-3}.$$

To obtain in explicit form the solution of the boundary-value problem for the rectangle D

$$\Delta_h u - 2\lambda u = f(x,y) \qquad u|_L = \beta(s), \quad (5)$$

where $\beta(s)$ is a given function on the boundary L of the rectangle D, it suffices (under the assumption that $\lambda' = 2\lambda$ is not an eigenvalue) to substitute

substitute the known quantities into formula (2)

$$A_k = \frac{1}{\Delta_k}\left\{\psi_k(x_{m+1})\hat{u}_k(x_0) - \psi_k(x_0)\left[\hat{u}_k(x_{m+1}) - \sum_{p=1}^{m} T^k(m+1-p)(\hat{h}^2\hat{f}_k(x_p) - \alpha^2\hat{\omega}_k(x_p))\right]\right\}, \tag{2'}$$

$$B_k = \frac{1}{\Delta_k}\left\{-\varphi_k(x_{m+1})\hat{u}_k(x_0) + \varphi_k(x_0)\left[\hat{u}_k(x_{m+1}) - \sum_{p=1}^{m} T_k(m+1-p)(\hat{h}^2\hat{f}_k(x) - \alpha^2\hat{\omega}_k(x_p))\right]\right\},$$

where $\{\hat{u}_k(x)\} = Su(x)$, $\{\hat{f}_k(x)\} = Sf(x)$, $\{\hat{\omega}_k(x)\} = S\vec{\omega}(x)$, $\Delta_k = \varphi_k(x_0)\psi_k(x_{m+1}) - \varphi_k(x_{m+1})\psi_k(x_0)$ $(k = 1, 2, \dots, n)$.

The eigenvalues $\lambda' = 2\lambda$ of the boundary-value problem (5) for the rectangle D are determined by the equalities

$$\lambda' = -4\left(\frac{1}{h^2}\sin^2\frac{i\pi}{2(m+1)} + \frac{1}{h_1^2}\cos^2\frac{k\pi}{2(n+1)}\right)$$

$$(i = 1, 2, \dots, m;\ k = 1, 2, \dots, n). \tag{6}$$

To solve the boundary-value problem (5) for a domain made up of the rectangle D and some other rectangle D_1, defined, for example, by the set of points (x_i, y_k) $(i = m, m+1, \dots, m+m'+1;\ k = n_0, n_0+1, \dots, n_0+n'+1 \le n;\ n_0 \ge 0)$, it is sufficient to write formula (2) for each of these rectangles and to equate the right-hand sides of the expressions obtained at the common points of D and D_1. To solve the boundary-value problem (5) for an arbitrary domain G bounded by a curvilinear contour Γ, one may inscribe in Γ a rectangle or some domain G' composed of several rectangles and, using formula (2), take as unknowns to be determined by numerical methods the values of u at the points of G that do not belong to G'. In exactly the same way, one obtains a sharp reduction in the number of algebraic equations subject to numerical solution in the case of boundary-value problems for equation (1) under various boundary conditions. Results analogous to those indicated here by the "method of summation relations" are obtained for the case of the Poisson equation with 3 independent variables, and also for the heat-conduction equation and for the wave equation.

In the case of the basic biharmonic problem, the values of u on the contour are prescribed, as well as linear relations between the pre-contour and contour values of u. Under such boundary conditions, formula (4) in the case of the rectangle D_2, defined by the set of points (x_i, y_k) $(i = 1, 2, \dots, m;\ k = 0, 1, \dots, n+1)$, directly reduces the solution of the biharmonic problem to the solution of $4n + 2m - 4$ linear algebraic equations, which, by means of simple transformations, reduce to $2m - 4$ equations. As in the case of the Poisson equation, formula

(4) makes it possible to obtain a sharp reduction in the number of algebraic equations subject to numerical solution when solving various boundary-value problems for the biharmonic equation in domains composed of a finite number of rectangles or bounded by curvilinear contours.

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Note: Figure translations are in progress. See original paper for figures.