THE STIELTJES INTEGRAL IN \(K\)-SPACES

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Abstract

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THE STIELTJES INTEGRAL IN K-SPACES

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In a previous paper (1) I established the general form of an additive bounded operator in the space of continuous functions with values in a semi-ordered normed space. In the present communication I establish an analogous theorem for the case of functions whose values belong to a semi-ordered space in which the existence of a real norm is not postulated. Whereas in the paper mentioned continuity was understood with respect to the norm, below we shall consider functions continuous with respect to order.

1. Stieltjes integrals. Let X be a K-space* and let $T = [a, b]$ be a segment of the real axis. We shall consider functions defined on T whose values belong to X. If such a function is bounded, then we denote

$$v(\delta) = \sup_{|t_1 - t_2| < \delta} |f(t_1) - f(t_2)|.$$

The function f is uniformly continuous (see also (3,4)) if

$$(o)\text{-}\lim_{\delta \to 0} v(\delta) = 0.$$

A function g, defined on T with values in a K-space Z, is of bounded variation on T if the set of all elements of the form $\sum_i |g(t_{i+1}) - g(t_i)|$, corresponding to all partitions of the segment T, is bounded. We denote by

$$\operatorname{var}_{t\in T} g(t)$$

or by W_g the least upper bound of this set.

Let Y also be a K-space and let $(X, Y)_o^r$ be the space of regular and (o)-continuous operators defined on X and with values in Y. In what follows we suppose that $Z = (X, Y)_o^r$.

Now let f be a uniformly (o)-continuous function on T with values in X, and let g be a function of bounded variation on T with values in Z. For any partition Δ of the segment T,

$$a = t_0 < t_1 < \dots < t_{\lambda-1} < t_\lambda = b,$$

we form the "Stieltjes-Riemann sum"

$$s = \sum_{i=0}^{\lambda-1} [g(t_{i+1}) - g(t_i)](f(\theta_i)), \tag{1}$$

where the "intermediate points" $\theta_i \in [t_i, t_{i+1}]$ are arbitrary.

We denote by δ the number equal to the greatest of the lengths of the partial segments in the partition Δ.

Let $\{\Delta_\nu\}$ be a sequence of partitions of the segment T such that $\delta_\nu \to 0$. For each ν denote by s_ν the sum of the form (1) constructed for the partition Δ_ν. If for all ν the partition $\Delta_{\nu+1}$ is a refinement

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[*] With some exceptions we shall use the notation of (^{2}).

of the partition Δ_ν, one may verify that

$$|s_\nu - s_{\nu+\mu}| \le W_g(v(\delta_\nu)) \qquad (\nu, \mu = 1, 2, \dots). \tag{2}$$

Since $W_g \in (X, Y)^r_o$, and $v(\delta_\nu) \to 0$, it follows from (2) that there exists $s = (o)\text{-}\lim\limits_\nu s_\nu$.

Then, as in the case of real functions, one may verify that for any sequence of partitions $\{\Delta'_\nu\}$ such that $\delta'_\nu \to 0$, and for any intermediate points used in the sums s'_ν, there exists $(o)\text{-}\lim\limits_\nu s'_\nu$, and it is equal to s. We call the element s the **integral of the function** f **with respect to the function** g and denote it by $\int_T dg(t)\, f(t)$.

It is easy to verify the usual properties:

$$\int_T dg(t)\, [f_1(t) + f_2(t)] = \int_T dg(t)\, (f_1(t)) + \int_T dg(t)\, (f_2(t)), \tag{3}$$

$$\int_T dg(t)\, [\lambda f(t)] = \lambda \int_T dg(t)\, f(t), \tag{4}$$

and also the relation

$$\left| \int_T dg(t)\, (f(t)) \right| \le W_g \left(\sup_{t \in T} |f(t)| \right). \tag{5}$$

2. Additive bounded operators. Denote by $M(T,X)$ the set of all bounded functions defined on the segment T, whose values belong to X. In this set we define, as usual, addition of elements and multiplication by numbers, and introduce the norm

$$|f| = \sup_{t\in T} |f(t)|$$

with values in X. It is not difficult to verify that $M(T,X)$ is a B_K-space (2). The subspace $C(T,X)$ of all uniformly continuous functions is also a B_K-space.

If $f_1, f_2 \in M(T,X)$, then we put $f_1 \leq f_2$ if $f_1(t) \leq f_2(t)$ for all $t \in T$. Then $M(T,X)$ and $C(T,X)$ are K-linear (2), and the modulus of an element $f \in C(T,X)$ coincides with the modulus of the same element computed in $M(T,X)$.

It is clear that

$$|f_1| \leq |f_2| \Rightarrow |f_1| \leq |f_2|.$$

If $\mathcal{V}$ is a space of type (B_K), normed by means of X, and U is an additive operator mapping $\mathcal{V}$ into a K-space Y, then we say that U is a **bounded operator** if there exists a positive operator $V \in (X,Y)^r_o$ such that

$$|U(f)| \leq V|f| \tag{6}$$

for all $f \in \mathcal{V}$.

In this case there exists a least operator V—the norm* of the operator U, denoted by $|U|$.

If $\mathcal{V} = C(T,X)$, then one may verify that** an additive operator U is bounded if and only if U is regular, and $|U|$ is (oo)-continuous on X. In this case $|U| = |U|$ on X.

* See (2), Ch. XII, 1.34. From $0 \leq |U| \leq V$ and $V \in (X,Y)^r_o$ it follows that $|U| \in (X,Y)^r_o$.

** We identify the set of constant functions in $C(T,X)$ with the set X.

For each $f \in C(T,X)$ denote

$$I(f) = \int_T dg(t)\,(f(t)),$$

where g is some function of bounded variation on T with values in $(X,Y)^r_o$.

From (3), (5) it follows that the operator I is additive and bounded, and

$$|I| \le \operatorname*{var}_{t\in T} g(t). \tag{7}$$

Let us now consider an arbitrary additive and bounded operator U on $C(T, X)$ with values in Y. We have

$$|U(f)| \le |U|(|f|),$$

where $|U| \in (X, Y)^r_o$. The operator U can be extended to $M(T, X)$ with preservation of additivity and norm (see (²), Ch. IX, 1.11). We shall denote the extended operator by the same letter.

For each $t \in T$ denote by $g(t)$ the operator from X into Y defined by the equality

$$g(t)(x) = U(\gamma_t x),$$

where, for $t \ne a$, γ_t is the characteristic function of the segment $[a, t]$, and $\gamma_a = 0$; $\gamma_t x$ is the function $(\gamma_t x)(\tau) = \gamma_t(\tau)(x)$. It is easy to see that for every $t \in T$ the operator $g(t)$ is regular and (oo)-continuous.

Since for all $x \ge 0$ $(x \in X)$ and for any partition Δ of the segment T

$$\left\{ \sum_i |g(t_{i+1}) - g(t_i)| \right\}(x) \le |U|(x),$$

it follows that g, as a function on T with values in $(X, Y)^r_o$, is of bounded variation and

$$\operatorname*{var}_{t\in T} g(t) \le |U|. \tag{8}$$

Let now $f \in C(T, X)$, and let Δ be an arbitrary partition of the interval T. In the notation of Section 1 we have

$$\sup_{t',t''\in[t_i,t_{i+1}]} |f(t') - f(t'')| \le v(\delta).$$

Put

$$h(t) = \begin{cases} f(t_0), & \text{for } a \le t \le t_1, \\ f(t_i), & \text{for } t_i < t \le t_{i+1} \ (i \ge 1). \end{cases}$$

It is clear that $h \in M(T, X)$ and

$$|f-h| \leq v(\delta),$$

whence it follows that

$$|U(f)-U(h)| \leq |U|(v(\delta)).$$

On the other hand,

$$h = \sum_i (\gamma_{t_{i+1}} - \gamma_{t_i}) f(t_i),$$

and, consequently,

$$U(h) = \sum_i [g(t_{i+1}) - g(t_i)](f(t_i)).$$

Hence we conclude that

$$U(f) = \int_T dg(t)\,(f(t)). \tag{9}$$

From (7) and (8) it also follows that

$$|U| = \operatorname*{var}_{t\in T} g(t). \tag{10}$$

We have obtained the following result:

Theorem. *The general form of an additive bounded operator transforming $C(T;X)$ into Y is given by formula (9), where g is a function of bounded variation with values in $(X,Y)^r_o$. In this case the function g can be chosen so that the equality (10) is satisfied.*

Remarks. 1°. If X is a regular K-space, and U is a regular operator on $C(T,X)$, then U is bounded. This assertion is obvious.

2°. Let $Y = X$, and let X be a K^+-space with unit (²). For $x \in X$ and $f \in C(T,X)$, denote by $x \cdot f$ the function $(x \cdot f)(t) = x \cdot f(t)$, if for every $t \in T$ the Boolean product $x \cdot f(t)$ exists and $\sup_{t\in T} |x \cdot f(t)| < \infty$.

Let U be an additive operator from $C(T,X)$ into X such that, if $x \cdot f$ exists, then

$$U(x \cdot f) = x \cdot U(f) \qquad (x \in X;\ f \in C(T,X));$$

we also assume that $U(f_\nu) \xrightarrow{o} 0$ if $|f_\nu| \xrightarrow{o} 0$. In this case U is bounded (see (^{4}), theorem 2), and we obtain Vulik' s result (see (^{4}), theorem 3).

3°. In the case when X is the real axis, and Y is a K^+-space of countable type, we obtain Kantorovich' s theorem (see (^{2}), Ch. VIII, 4.11).

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Note: Figure translations are in progress. See original paper for figures.