MATHEMATICS

1960

SovietRxiv

Abstract

Full Text

MATHEMATICS

M. A. KRASNOSEL′SKII

REGULAR AND COMPLETELY REGULAR CONES

(Presented by Academician P. S. Aleksandrov on 15 VI 1960)

1. In many constructions of linear and nonlinear functional analysis one has to establish the convergence of sequences of elements that are monotone in the sense of a certain semi-ordering. In connection with this it is natural to introduce the following definitions.

A cone K in a real Banach space E will be called **regular** if every sequence $x_n \in E$ satisfying the conditions

$$x_1 \le x_2 \le \cdots \le x_n \le \cdots, \tag{1}$$

$$x_n \le z \quad (n = 1, 2, \ldots), \tag{2}$$

converges in norm to some element of the space E. A cone K will be called **completely regular** if every sequence $x_n \in E$ satisfying condition (1) and bounded in norm converges in norm:

$$\|x_n\| \le M \quad (n = 1, 2, \ldots). \tag{3}$$

As is known, (3) does not follow from (2), and from (3) there does not follow the existence of such a z that condition (2) is fulfilled. However, the following assertion holds:

Theorem 1. *Every completely regular cone is regular.*

Let K_+ denote the cone of nonnegative functions. In the spaces L^p, $1 \le p < \infty$, and in the Orlicz spaces L_M, where $M(u)$ satisfies the Δ_2-condition, the cone K_+ is completely regular. In the space C and in the spaces L_M^*, where $M(u)$ does not satisfy the Δ_2-condition, the cone K_+ does not even possess the property of regularity. In the space E_M (the closure in L_M^* of the set of bounded functions) the cone K_+ is regular, but not completely regular, if $M(u)$ does not satisfy the Δ_2-condition.

Obviously, regularity implies complete regularity if the cone is solid, i.e. contains interior points. Regularity implies complete regularity if the space E is weakly complete (this fact, relying on certain results of M. G. Krein, was established by V. Ya. Stetsenko).

2. According to M. G. Krein, a cone K is called **normal** if there exists a $\delta > 0$ such that for any $e, g \in K$, from $\|e\| = \|g\| = 1$ there follows the inequality $\|e + g\| > \delta$.

Theorem 2. *Every regular cone is normal.*

The converse assertion is false, since the cone of nonnegative functions in the space C is normal, but, as we have already noted, does not possess the property of regularity.

3. Let u be a fixed nonzero element of K. Denote by E_u the collection of those $x \in E$ for which, for some $a = a(x)$, the inequalities

$$-au \leq x \leq au.$$

The least of the numbers a for which these inequalities are satisfied is called the u-norm of the element x and is denoted by $\|x\|_u$.

The following criterion for normality of a cone seems convenient to us (cf. D. P. Mil’ man’ s criterion, given in (^{2}), and I. A. Bakhtin’ s criterion (^{4})).

Theorem 3. *For the cone K to be normal it is necessary and sufficient that there exist an $M > 0$ such that, for every $y \in K$, the inequality*

$$\|x\| \leq M\|y\|\|x\|_y \qquad (x \in E_y) \tag{4}$$

holds.

4. A positive (not necessarily linear) functional $f(x)$, defined on K, will be called **strictly increasing** if, for arbitrary $h_n \in K$ $(n = 1, 2, \dots)$, from

$$\|h_n\| \geq \varepsilon_0 > 0 \qquad (n = 1, 2, \dots)$$

it follows that

$$\lim_{n\to\infty} f(h_1 + \dots + h_n) = \infty.$$

In the spaces L_p $(1 \leq p < \infty)$, a strictly increasing functional can be defined, for example, by the equality $f(x) = \|x\|^p$.

Theorem 4. *Let a functional strictly increasing on the cone K be bounded on the intersection of the cone with every ball of the space E.*

Then the cone K is completely regular.

Theorem 5. *Let it be possible to define on the cone K a monotone strictly increasing functional.*

Then the cone K is regular.

5. A linear functional $f(x)$ will be called **uniformly positive** if

$$f(x) \geq a\|x\| \qquad (x \in K), \tag{5}$$

where a is some positive number. For linear positive functionals, uniform positivity is equivalent to the fact that the functional is strictly increasing.

We shall say that a cone K **admits plastering** if one can specify a cone K_1 such that every nonzero element $x \in K$ is an interior element of the cone K_1 and, moreover, lies in the cone K_1 together with the spherical neighborhood of radius $b\|x\|$, where b does not depend on x.

Theorem 6. *In order that the cone K admit plastering, it is necessary and sufficient that a uniformly positive linear functional can be defined on it.*

From this theorem and from Theorem 6 it follows that

Theorem 7. *Every cone admitting plastering is completely regular.*

The cone of nonnegative functions in the space L admits plastering. The cones of nonnegative functions in the spaces L_p $(1 < p < \infty)$ do not admit plastering. Thus, not every completely regular cone admits plastering.

If the cone K is solid, then the conjugate cone K^* in the space of linear functionals admits plastering, since on K^* a uniformly positive linear functional can be defined as the value at a fixed interior element of the cone K. Therefore the cone K^*, conjugate to a solid cone K, is completely regular (this fact was obtained by V. Ya. Stetsenko by other considerations).

6. We give two important examples of cones admitting plastering.

Let F be a bounded closed and convex set not containing the zero θ of the space E. Denote by $K(F)$ the totality of all elements $x \in E$ admitting a representation $x = tz$, where $t \geqslant 0$ and $z \in F$. The set $K(F)$ is a cone admitting faceting.

Let u be a fixed nonzero element of K; let $\rho \geqslant 1$. Denote by $K_{u;\rho}$ (cf. $(1, ^3)$) the totality of all such elements $x \in K$ that

$$au \leqslant x \leqslant a\rho u,$$

where $a = a(x) \geqslant 0$. It is not hard to see that $K_{u;\rho}$ is a cone; this cone admits faceting if the cone K is normal.

7. A strengthening of the second part of Theorem 6 is

Theorem 8. *Let a linear completely continuous (on K) operator A be given on the cone K, satisfying the condition*

$$\|Ax\| \geqslant a\|x\| \quad (x \in K). \tag{6}$$

Then the cone K admits faceting.

8. We shall call a cone K **locally compact** if its intersection with every ball is a compact set. It follows from Theorem 8 that locally compact cones admit faceting—it suffices to consider the identity operator on the cone.

9. In conclusion we give one theorem on linear operators leaving invariant a certain cone K. An operator A is called (see ([3])) u_0**-positive** if there exists a nonzero element $u_0 \in K$ such that for every nonzero $x \in K$ one can indicate a natural number $p = p(x)$ and positive numbers $\alpha = \alpha(x)$, $\beta = \beta(x)$ such that

$$\alpha u_0 \leqslant A^p x \leqslant \beta x.$$

Theorem 9. *Let a linear operator A leave invariant a reproducing cone K and be a u_0-positive operator. Let φ_0 be a positive eigenvector of the operator A, not belonging to the invariant subspace Π of the operator A.*

Finally, suppose that one of the three conditions is fulfilled: a) the space Π is finite-dimensional; b) the operator A is completely continuous; c) the subspace Π is weakly complete and weakly compact, and the cone $K \cap \Pi$ admits faceting.

Then the intersection of the invariant subspace Π with the cone K consists of the single zero point θ.

Voronezh State
University

Received
11 VI 1960

References

[1] M. G. Krein, S. G. Krein, *Matem. sborn.*, **13** (55), issue 1, 1 (1943).
[2] M. G. Krein, M. A. Rutman, *Uspekhi Mat. Nauk*, **3**, 1 (23) (1948).
[3] M. A. Krasnosel' skii, *Topological Methods in the Theory of Nonlinear Integral Equations*, Moscow, 1956.
[4] I. A. Bakhtin, *Proceedings of the Seminar on Functional Analysis*, Voronezh, issue 5 (1959).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.