SOLUTION OF THE DIRICHLET PROBLEM IN THE PLANE FOR A SECOND-ORDER EQUATION OF ELLIPTIC TYPE BY THE METHOD OF EXPANSION IN A SERIES

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Abstract

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SOLUTION OF THE DIRICHLET PROBLEM IN THE PLANE FOR A SECOND-ORDER EQUATION OF ELLIPTIC TYPE BY THE METHOD OF EXPANSION IN A SERIES

(Presented by Academician I. N. Vekua, 10 II 1960)

1. In the paper (1) a method was indicated for solving the generalized Riemann–Hilbert problem, using a certain system of particular solutions of the adjoint equation. In the present paper, by the same method, the Dirichlet problem (problem D) in the plane will be studied for a second-order elliptic equation with (real) analytic coefficients (2).
2. For simplicity we shall restrict ourselves to the consideration of a simply connected domain, since without particular difficulty the propositions given below can be extended to the case of a multiply connected domain.

Problem D. It is required to find a solution $u(x,y)$, regular in T and continuous in $T+\Gamma$, of the equation

$$\mathfrak{M}u = \Delta u + a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} + c(x,y)u = 0, \tag{1}$$

satisfying the boundary condition

$$u(t)\big|_{\Gamma} = \gamma(t), \qquad t \in \Gamma, \tag{2}$$

where the coefficients a, b, c are real analytic functions in some fundamental domain $\mathfrak{D}$ of equation (1) (2). Suppose that the domain T lies inside the fundamental domain $\mathfrak{D}$, and that its boundary Γ is a closed H-continuous curve.

If $u(x,y)$ is a regular solution of equation (1), then (2)

$$u(x,y) = \int_{\Gamma}\left[u(t)N\omega(x,y,\xi,\eta) - \frac{du(t)}{d\nu}\omega(x,y,\xi,\eta)\right]ds, \tag{3}$$

where $N\omega = d\omega/d\nu - [a\cos(\nu,x) + b\cos(\nu,y)]\omega$; ν is the inward normal at the point (ξ,η); ω is the normalized standard elementary solution of equation (1).

It is obvious that formula (3) will allow us to obtain the solution of problem D, if the boundary values $du(t)/d\nu$ on the boundary Γ are known ($u(t)$ is already prescribed). It remains to determine the boundary values $du(t)/d\nu$. For this purpose we use a certain complete system $\{v_k(x, y)\}$ of particular solutions of the adjoint equation*

$$\mathfrak{N}v \equiv \Delta v - \frac{\partial av}{\partial x} - \frac{\partial bv}{\partial y} + cv = 0. \tag{4}$$

* Completeness here is understood in the sense that every solution of equation (4) regular in T can be uniformly approximated inside T by means of linear combinations of the form $\xi_1 v_1 + \xi_2 v_2 + \dots + \xi_{nv} n$ (^{2}).

Every solution v of this equation can be represented in the form [(2)]

$$v(x, y) = \operatorname{Re}\left[H_0(z)\varphi(z) + \int_0^z H(z, \tau)\varphi(\tau)\, d\tau\right], \tag{5}$$

where $\varphi(z)$ is an arbitrary holomorphic function. Substituting into (5), in place of $\varphi(z)$, the system z^p, iz^p ($p = 0, 1, 2, \ldots$), we obtain one complete (infinite) system $\{v_k(x, y)\}$ of particular solutions of equation (4) with respect to the domain T.

Moreover, for any solution v of equation (4) and solution u of problem D, the equality

$$\iint_T [v\mathfrak{M}u - u\mathfrak{M}v]\, dx\, dy = \int_\Gamma \left[\gamma(t) N v(t) - \frac{du(t)}{d\nu} v(t)\right] ds = 0 \tag{6}$$

holds. This is, obviously, equivalent to

$$\int_\Gamma \frac{du(t)}{d\nu} v_k(t)\, ds = \int_\Gamma \gamma(t) N v_k(t)\, ds \equiv c_k, \qquad k = 1, 2, \ldots, \tag{7}$$

where c_k are known constants.

3. We begin the construction of the solution of problem D with the case when the homogeneous problem D_0 has no solution. It is not difficult to prove the following theorem:

Theorem 1. If the homogeneous problem D_0 has no solution, then on the contour Γ the system $v_k(t)$, $k = 1, 2, \ldots$, is linearly independent; moreover, as is known [(2)], it is closed in the space $L^2(\Gamma)$, i.e., for any $\varepsilon > 0$ and any function $f(t)$ belonging to the space $L^2(\Gamma)$, there is a system of constants $\xi_1, \xi_2, \ldots, \xi_n$ such that

$$\left\| f(t) - \sum_{k=1}^{n} \xi_k v_k(t) \right\|_{L^2} < \varepsilon.$$

Thus, in this case one may assume that on Γ the system $v_k(t)$, $k = 1, 2, ...$, is complete and orthonormalized in the space $L^2(\Gamma)$. Consequently, according to formula (7), we obtain that the Fourier series $\sum_{k=1}^{\infty} c_k v_k(t)$ on the contour Γ converges in mean to the function $du(t)/d\nu$. Hence the following theorem follows:

Theorem 2. If the homogeneous problem D_0 has no solution, then in the domain T, for the solution of the Dirichlet problem, the equality

$$u(x, y) = \int_\Gamma \left[\gamma(t) N\omega(x, y, \xi, \eta) - \sum_{k=1}^{\infty} c_k v_k(t) \omega(x, y, \xi, \eta) \right] ds, \tag{8}$$

holds, and the right-hand side of (8) converges uniformly to the solution $u(x, y)$ in the closed domain $T + \Gamma$.

4. We proceed to consider the case when the homogeneous problem D_0 has nontrivial solutions. As is known [(2)], the number of linearly independent solutions of problem D_0 is always finite, and the adjoint problem D_0^* has as many linearly independent solutions as problem D_0. Suppose that $u_1, u_2, \dots, u_l$ is some complete system of solutions of problem D_0. In this case, by a suitable choice of the system $\{v_k(x, y)\}$, $k = 1, 2, ...$, it is always possible to ensure that on the contour Γ, $v_1(t) = \dots = v_l(t) = 0$, and that the system $\{v_{l+k}(t)\}$, $k = 1, 2, ...$, is orthonormal.

According to formula (7),

$$\int_\Gamma \gamma(t) \frac{dv_k(t)}{dv} \, ds = 0, \qquad k = 1, 2, \dots, l. \tag{9}$$

These are the solvability conditions for problem D.

Consider the space of all solutions of problem D_0 (an l-dimensional space). For each element $\tilde{u}(x, y) \not\equiv 0$ of this space we have $d\tilde{u}(t)/dv \not\equiv 0$, $t \in \Gamma$, since otherwise, according to formula (3), it would follow that $\tilde{u}(x, y) \equiv 0$ in T. All boundary values $d\tilde{u}(t)/dv$, $t \in \Gamma$, form an l-dimensional subspace R of the space $C(\Gamma)$. It is easy to see that the linearly independent system $du_k(t)/dv$, $k = 1, 2, \dots, l$, may be taken as a basis of the subspace R.

If $f(t) \in L^2(\Gamma)$, then for any $\varepsilon > 0$ there exists a continuous function $g(t)$ such that $\|f(t) - g(t)\|_{L^2} < \varepsilon/2$; on the other hand, the continuous function $g(t)$ can be uniquely represented in the form $g(t) = g_1(t) + g_2(t)$, where $g_1(t)$ belongs to the subspace R, i.e.

$$g_1(t) = \sum_{k=1}^{l} \eta_k \frac{du_k(t)}{dv},$$

and $g_2(t)$ belongs to the orthogonal complement to R, i.e. for $g_2(t)$ we have

$$\int_\Gamma g_2(t) \frac{du_k(t)}{dv}\, ds = 0, \qquad k = 1, 2, \dots, l. \tag{10}$$

This is nothing other than the solvability condition for the adjoint problem D^*, so that one can find a solution $v(x, y)$ of problem D^*, and $v(t)|_\Gamma = g_2(t)$. It is easy to see that in this case the holomorphic function $\varphi(z)$, by means of which $v(x, y)$ is expressed in the form (5), belongs (at least) to the space $L^2(T + \Gamma)$ ([3]), therefore $\varphi(t)$, $t \in \Gamma$, can be approximated in the mean by a polynomial on Γ; consequently, there exists a system of constants $\xi_1, \xi_2, \dots, \xi_n$ such that on Γ the inequality

$$\left\| g_2(t) - \sum_{k=1}^{n} \xi_k v_k(t) \right\|_{L^2} < \frac{\varepsilon}{2},$$

holds; whence it follows that

$$\left\| f(t) - \sum_{k=1}^{l} \eta_k \frac{du_k(t)}{dv} - \sum_{k=1}^{n} \xi_k v_k(t) \right\|_{L^2} \leq$$

$$\leq \|f(t) - g(t)\|_{L^2} + \left\| g(t) - \sum_{k=1}^{l} \eta_k \frac{du_k(t)}{dv} - \sum_{k=1}^{n} \xi_k v_k(t) \right\|_{L^2} < \varepsilon.$$

Theorem 3. If the homogeneous problem D_0 has l linearly independent solutions $u_1, u_2, \dots, u_l$, then the system

$$\frac{du_1(t)}{dv},\ \frac{du_2(t)}{dv}, \dots, \frac{du_l(t)}{dv},\ v_{l+1}(t),\ v_{l+2}(t), \dots, \tag{11}$$

will be closed in the space $L^2(\Gamma)$.

Thus, one may assume that the system (11) is a complete orthonormal system in the space $L^2(\Gamma)$. Under conditions (9), if we put

$$a_k = \int_\Gamma \frac{du(t)}{dv} \frac{du_k(t)}{dv}\, ds, \qquad k = 1, 2, \dots, l,$$

then the Fourier series $\sum_{k=1}^{l} a_k \frac{du_k(t)}{dv}+$

$$+\sum_{k=1}^{\infty} c_{l+k} v_{l+k}(t)$$

converges in the mean to the function $\frac{du(t)}{d\nu}$. In particular, the series

$$\sum_{k=1}^{\infty} c_{l+k} v_{l+k}(t)$$

converges in the mean to the function

$$\frac{d}{d\nu}\left(u(t)-\sum_{k=1}^{l} a_k u_k(t)\right).$$

If we denote

$$u^*(x,y) \equiv u(x,y)-\sum_{k=1}^{l} a_k u_k(x,y),$$

then, evidently, $u^*(x,y)$ is a certain particular solution of problem D. Hence it follows that

$$\int_\Gamma \left[\gamma(t) N\omega(x,y,\xi,\eta)-\sum_{k=1}^{\infty} c_{l+k} v_{l+k}(t)\,\omega(x,y,\xi,\eta)\right] ds$$

converges uniformly to the particular solution $u^*(x,y)$ in the closed domain $T+\Gamma$. After this, the general solution of problem D in the domain T can be obtained from the formula

$$u(x,y)=u^*(x,y)+\sum_{k=1}^{l} \lambda_k u_k(x,y),$$

where λ_k are arbitrary constants.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.