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M. G. KREIN

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Abstract

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M. G. KREIN

ON THE THEORY OF LINEAR NON-SELF-ADJOINT OPERATORS

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In what follows, $\mathfrak{H}$ denotes a separable Hilbert space; $\mathfrak{R}$ is the linear ring of all linear bounded operators acting in $\mathfrak{H}$; $\mathfrak{J}$ is the two-sided ideal in $\mathfrak{R}$ of all completely continuous operators; $\mathfrak{S}$ is the two-sided ideal in $\mathfrak{R}$ of all operators $A \in \mathfrak{J}$ such that $\operatorname{Sp}(A^*A)^{1/2} < \infty$ (i.e., having trace $\operatorname{Sp} A$ absolutely convergent).

If $A \in \mathfrak{S}$, then the quantity $\det(I + A)$ is meaningful, defined as the limit, as $n \to \infty$, of the determinants $\left|\delta_{jk} + (A\varphi_j, \varphi_k)\right|_1^n$, where $\{\varphi_j\}_1^\infty$ is any orthonormal basis in $\mathfrak{H}$; this limit always exists and does not depend on the choice of basis. It is easily shown that if $A, B \in \mathfrak{J}$, $AB \in \mathfrak{S}$ and $BA \in \mathfrak{S}$, then $\det(I + AB) = \det(I + BA)$.

If $A, B \in \mathfrak{R}$, $B - A \in \mathfrak{S}$, then the determinant

$$D_{B/A}(\lambda) = \det[(I - \lambda B)(I - \lambda A)^{-1}] = \det[I + \lambda(B - A)(I - \lambda A)^{-1}]$$

is meaningful for all complex λ for which $(I - \lambda A)^{-1} \in \mathfrak{R}$ exists. In (1) it was already noted that if $A \in \mathfrak{R}$, $B - A \in \mathfrak{S}$, $C - B \in \mathfrak{S}$, then $D_{C/A}(\lambda) = D_{C/B}(\lambda) D_{B/A}(\lambda)$.

An operator $A \in \mathfrak{R}$ is called **dissipative** if, in its decomposition into Hermitian components $A = A_R + iA_J$, the imaginary component $A_J = (A - A^*)/2i$ is a nonnegative operator: $(A_J f, f) \geq 0$ $(f \in \mathfrak{H})$.

The following generalization of a theorem of M. S. Livshits (^{2}) and M. S. Brodskii (^{3}) on the characteristic operator of a dissipative operator holds.

Theorem 1. *Let* $A = G + iH$ $(H = A_J)$ *be a dissipative operator,* $B = G + iF$*, where* $-H \leq F \leq H$*.*

Then, for $\operatorname{Im} \lambda < 0$*, the operator*

$$W_\lambda = I + i(H - F)^{1/2}(A - \lambda I)^{-1}(H - F)^{1/2}$$

is nonexpanding, i.e. $\|W_\lambda f\| \leq \|f\|$ $(f \in \mathfrak{H})$*.*

The theorem follows from the easily verified identity

$$I - W_\lambda^* W_\lambda = (H-F)^{1/2}[R_\lambda^*(H+F)R_\lambda - 2\operatorname{Im}\lambda R_\lambda^* R_\lambda](H-F)^{1/2},$$

where $R_\lambda = (A-\lambda I)^{-1}$. If, in addition, the condition $H \in \mathfrak{S}$ is fulfilled (i.e. $\operatorname{Sp} H < \infty$), then, putting $T = A - B = H - F$, we shall have

$$D_{B/A}(\lambda) = \det[I - i\lambda T(I-\lambda A)^{-1}] = \det[I - i\lambda T^{1/2}(I-\lambda A)^{-1}T^{1/2}] = \det W_{1/\lambda}.$$

Theorem 2. *If the operators A and B satisfy the conditions of the preceding theorem and* $\operatorname{Sp} H < \infty$*, then* $|D_{B/A}(\lambda)| \leq 1$ *for* $\operatorname{Im}\lambda > 0$*.*

If $A = G + iH \in \mathfrak{R}$, $H \in \mathfrak{S}$, then, putting $A_1 = G + iH_1$, $H_1 = H_+ + H_-$, where H_+ and H_- are the orthogonal nonnegative operators from the decomposition $H = H_+ - H_-$, we shall have

$$D_{G/A}(\lambda) = D_{G/A_1}(\lambda) D_{A_1/A}(\lambda) = D_{G/A_1}(\lambda)/D_{A/A_1}(\lambda),$$

where, by Theorem 2, the functions $D_{G/A_1}(\lambda)$ and $D_{A/A_1}(\lambda)$ will have modulus $\leqslant 1$ for $\operatorname{Im}\lambda > 0$. Hence

Theorem 3. *If $A = G + iH \in \mathfrak{R}$, $H \in \mathfrak{S}$, then inside the upper (lower) half-plane* $\operatorname{Im}\lambda > 0$ ($\operatorname{Im}\lambda < 0$) *the function $D_{G/A}(\lambda)$ can be represented as a quotient of two holomorphic bounded functions.*

2. By the multiplicity of an x-number (characteristic number) of an operator $A \in \mathfrak{S}$ is meant the dimension of the corresponding root subspace of the operator A. By $n(r;A)$, $n_\pm(r;A)$ are denoted, respectively, the exact number (i.e., counting multiplicity) of the x-numbers of the operator A in the circle $|\lambda| \leqslant r$, in the interval $(0, r]$ or $[-r, 0]$.

An operator $A \in \widetilde{\mathfrak{S}}$ is called **Volterra** if it has no x-numbers.

A number of assertions of Theorem 4 below are easily derived from Theorem 3 and from a theorem of the author (^{4}), according to which an entire function $f(\lambda)$, representable inside each of the two half-planes $\operatorname{Im}\lambda > 0$ and $\operatorname{Im}\lambda < 0$ as a quotient of two bounded holomorphic functions, always has the properties:

$$1) \quad \ln|f(\lambda)| = O(|\lambda|) \quad \text{as } \lambda \to \infty; \qquad 2) \quad \int_{-\infty}^{\infty} \frac{|\ln|f(\lambda)||}{1+\lambda^2}\, d\lambda < \infty. \quad (1)$$

Theorem 4. *If the operator $A = G + iH$ is Volterra and $H \in \mathfrak{S}$, then the entire function $f(\lambda) = D_{G/A}(\lambda)\exp(-i\lambda \operatorname{Sp} H)$ has the properties (1) and is representable in the form*

$$f(\lambda) = \prod_j (1 - \lambda/a_j) e^{\lambda/a_j},$$

where $\{a_j\}$ *is the complete sequence of x-numbers of the operator* A_**. For this sequence there exists the common limit*

$$\frac{h}{\pi} = \lim_{r\to\infty} \frac{n_+(r;G)}{r} = \lim_{r\to\infty} \frac{n_-(r;G)}{r},$$

where $|\operatorname{Sp} H| \leqslant h \leqslant \operatorname{Sp}|H|$ $(= \operatorname{Sp} H_+ + \operatorname{Sp} H_-)$*. If, in particular, the operator* A *is dissipative, then* $h = \operatorname{Sp} H$*.*

A weaker assertion was formulated in (^{5}). The general method set forth in (^{6}) (see also (^{1})) makes it possible to draw the following conclusion from Theorem 4.

Theorem 5. *If the operator* $A = G + iH \in \mathfrak{S}$ *is dissipative,* $\operatorname{Sp} H < \infty$*, and at least one of the two conditions is satisfied:*

$$1) \quad \lim_{r\to\infty} \frac{n_+(r;G)}{r} = 0; \qquad 2) \quad \lim_{r\to\infty} \frac{n_-(r;G)}{r} = 0,$$

then the system of root vectors of the operator A *is complete in* $\mathfrak{H}$*.*

This theorem, being a strengthening of Theorem 1 from (^{1}), in essence already follows from the considerations given in the present article.

3. If $A = G + iH \in \mathfrak{S}$ and $H \in \mathfrak{S}$, then

$$D_{A^*/A}(\lambda) = D_{A^*/G}(\lambda) D_{G/A}(\lambda) = D_{G/A}(\lambda)/\overline{D}_{G/A}(\lambda). \tag{2}$$

Let us explain that we write $g(\lambda) = \overline{f}(\lambda)$ if $g(\bar{\lambda}) = \overline{f(\lambda)}$. From (2) it follows that $|D_{A^*/A}(\lambda)| = 1$ for $\operatorname{Im}\lambda = 0$. Hence, from Theorem 3, one obtains without difficulty:

Theorem 6. *If the operator* $A = G + iH \in \mathfrak{S}$ *and* $H \in \mathfrak{S}$*, then*

$$D_{A^*/A}(\lambda) = e^{2ia\lambda} \prod_j \frac{1 - \lambda/\bar{\lambda}_j}{1 - \lambda/\lambda_j},$$

where $\{\lambda_j\}$ is the complete sequence of characteristic values of the operator A,

$$a = \operatorname{Sp} H - \sum_j \operatorname{Im}\left(\frac{1}{\lambda_j}\right). \tag{3}$$

4. As is known, the logarithmic length L_g of a measurable set $\Delta \subset (1, \infty)$ is the integral over Δ of dr/r. In the paper (^{7}) W. K. Hayman established an important theorem which, in particular, contains the following proposition.

Let $u(\lambda)$ be a nonnegative superharmonic function in the open upper half-plane, and let

$$h = \inf(u(\lambda)/\operatorname{Im}\lambda),$$

where λ ranges over this half-plane. Then there exists a set $\Delta \subset (1, \infty)$ with $L_g(\Delta) < \infty$ such that, as $\rho \to \infty$, outside Δ, uniformly in θ $(0 < \theta < \pi)$, the limiting relation

$$\lim(u(\rho e^{i\theta})/\rho) = h \sin\theta$$

will hold.

This proposition, in combination with Theorems 3 and 6, and also with Jensen' s formula:

$$N(\rho; G) - N(\rho; A) = \int_0^\rho \frac{n(r; G)}{r}\, dr - \int_0^\rho \frac{n(r; A)}{r}\, dr = \frac{1}{2\pi}\int_0^{2\pi} \ln|D_{G/A}(\rho e^{i\theta})|\, d\theta$$

leads to the following conclusion:

Theorem 7. If $A = G + iH \in \mathfrak{S}$, $H \in \mathfrak{S}$, then, as $\rho \to \infty$, outside a suitable set Δ with $L_g(\Delta) < \infty$, the asymptotic relation

$$N(\rho; G) - N(\rho; A) = c\rho + o(\rho), \tag{4}$$

holds, where the constant $c \geq 2|a|/\pi$ (the quantity a is defined in (3)). If the operator A is dissipative, then $c = 2a/\pi$.

Hence, and from a theorem of M. S. Livshits (^{2}) (see also (^{8}), it follows that

Theorem 8. In order that the system of root vectors of a completely continuous dissipative operator $A = G + iH$ with $H \in \mathfrak{S}$ be complete, it is necessary and sufficient that

$$N(\rho; G) - N(\rho; A) = o(\rho),$$

as $\rho \to \infty$, outside a suitable set Δ with $L_g(\Delta) < \infty$.

It should be pointed out that in his recent paper (^{9}) B. Ya. Levin developed methods that allowed him to show (without using infinite determinants) that, in the case of a dissipative operator $A = G + iH \in \mathfrak{S}$ with $\operatorname{Sp} H < \infty$, the asymptotic inequality

$$N(\rho; G) - N(\rho; A) \leq \rho \operatorname{Sp} H + o(\rho)$$

holds as $\rho \to \infty$, outside some set Δ with $L_g(\Delta) < \infty$. A joint discussion of the considerations of paper (^{9}) showed that a little must be added to them in order, for the indicated case, to obtain relation (4) with some $c \leq 2a/\pi$ (for this case Theorem 7 asserts that $c = 2a/\pi$). After this the author reexamined the methods of his paper (1) and found that, after supplementing them somewhat, one can arrive at the results of the present paper.

Odessa Civil Engineering Institute

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CITED LITERATURE

[1] M. G. Krein, *Uspekhi Mat. Nauk*, **14**, no. 3 (87), (1959).
[2] M. S. Livshits, *Mat. Sb.*, **34** (76), 1 (1954).
[3] M. S. Brodskii, *Mat. Sb.*, **39** (81), 2 (1956).
[4] M. G. Krein, *Izv. AN SSSR, Ser. Mat.*, **11**, 309 (1947).
[5] I. Ts. Gohberg, M. G. Krein, *DAN*, **128**, no. 2 (1959).
[6] V. B. Lidskii, *DAN*, **119**, no. 6 (1958); *Tr. Moskov. Mat. Obshch.*, **8**, 83 (1959).
[7] W. K. Hayman, *J. Math. Pures et Appl.*, **35**, 115 (1956).
[8] B. R. Mukminov, *DAN*, **99**, no. 4 (1954).
[9] B. Ya. Levin, *Sb. Tr. Kharkov. Inst. Inzh. Zh.-D. Transporta*, 1959.

Note: Figure translations are in progress. See original paper for figures.

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