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Abstract

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ON THE DECOMPOSITION INTO IRREDUCIBLE REPRESENTATIONS OF THE TENSOR PRODUCT OF TWO REPRESENTATIONS OF THE COMPLEMENTARY SERIES OF THE PROPER LORENTZ GROUP

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In previous papers ([4–6]) the author solved the problem of decomposing into irreducible representations the tensor product of two irreducible unitary representations of the proper Lorentz group in the two cases when both factors belong to the principal series, or when one of these representations belongs to the principal series and the other to the complementary series.

The present paper is devoted to the remaining case, when both factors belong to the complementary series. Namely, we determine here into which irreducible representations the tensor product $\mathfrak{D}_{\nu_1} \times \mathfrak{D}_{\nu_2}$ of two representations $\mathfrak{D}_{\nu_1}, \mathfrak{D}_{\nu_2}$ of the complementary series decomposes. Throughout this paper the notation and results of papers ([4],[5]) are used.

The representation $\mathfrak{D}_{\nu_1} \times \mathfrak{D}_{\nu_2}$ is realized in the space $\mathfrak{H}_{\nu_1} \times \mathfrak{H}_{\nu_2}$—the completion with respect to the scalar product

$$(f_1, f_2) = \int |z_1 - z_1'|^{-2+\nu_1} |z_2 - z_2'|^{-2+\nu_2} f_1(z_1, z_2) \overline{f_2(z_1', z_2')} \, dz_1 \, dz_2 \, dz_1' \, dz_2'$$

of the space $\mathfrak{H}'_{\nu_1} \times \mathfrak{H}'_{\nu_2}$ of all measurable functions $f(z_1, z_2)$ for which

$$\int |z_1 - z_1'|^{-2+\nu_1} |z_2 - z_2'|^{-2+\nu_2} |f(z_1, z_2)| \, |f(z_1', z_2')| \, dz_1 \, dz_1' \, dz_2 \, dz_2' < \infty;$$

the operators T_g of the representation $\mathfrak{D}_{\nu_1} \times \mathfrak{D}_{\nu_2}$ on $f \in \mathfrak{H}'_{\nu_1} \times \mathfrak{H}'_{\nu_2}$ are given by the formula

$$T_g f(z_1, z_2) = |\beta z_1 + \delta|^{-\nu_1 - 2} |\beta z_2 + \delta|^{-\nu_2 - 2} f\left(\frac{\alpha z_1 + \gamma}{\beta z_1 + \delta}, \frac{\alpha z_2 + \gamma}{\beta z_2 + \delta} \right) \qquad (1)$$

$$\left(\text{for } g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right)$$

and are then extended by continuity to all of $\mathfrak{H}_{\nu_1} \times \mathfrak{H}_{\nu_2}$. The spaces $\mathfrak{H}_\nu$, $\mathfrak{H}_{\nu_1} \times \mathfrak{H}_{\nu_2}$ can also be described as follows.

Let $K(Z)$ be the set of all infinitely differentiable finite functions of z (i.e., of $x = \operatorname{Re} z,\ y = \operatorname{Im} z$). Clearly, $K(Z)$ is dense in $\mathfrak{H}'_\nu$, and hence also in $\mathfrak{H}_\nu$. The Fourier transform

$$\widetilde{\varphi}(w) = \frac{1}{2\pi} \int \varphi(z) e^{i \operatorname{Re}(z\overline{w})}\, dz$$

is an isometric mapping $F : \varphi \to \widetilde{\varphi}$ of the set $K(Z) \subset \mathfrak{H}_\nu$ onto a dense subset of the Hilbert space $\widetilde{\mathfrak{H}}_\nu$ of all measurable func-

functions $\widetilde{\varphi}(w)$ for which $\int |w|^{-\nu} |\widetilde{\varphi}(w)|^2\, dw < \infty$, with scalar product

$$(\widetilde{\varphi}_1, \widetilde{\varphi}_2) = a(\nu) \int |w|^{-\nu} \widetilde{\varphi}_1(w) \overline{\widetilde{\varphi}_2(w)}\, dw,$$

where

$$a(\nu) = 2^\nu \pi \Gamma\left(\frac{\nu}{2}\right) \left[\Gamma\left(1 - \frac{\nu}{2}\right)\right]^{-1}$$

(see (3), pp. 2 and 3, § 12). Consequently, F extends uniquely to an isometric mapping, which we again denote by F, of the whole space $\mathfrak{H}_\nu$ onto $\widetilde{\mathfrak{H}}_\nu$. Therefore every element $f \in \mathfrak{H}_\nu$ may be realized as a generalized function on $K(Z)$, putting

$$(f; \varphi) = \int \tilde{f}(w) \widetilde{\varphi}(w)\, dw$$

for $\varphi \in K(Z)$, $\tilde{f} = Ff$, $\widetilde{\varphi} = F\varphi$. It follows from this that $\mathfrak{H}_\nu$ consists of precisely those generalized functions on $K(Z)$ whose Fourier transforms are ordinary functions belonging to $\widetilde{\mathfrak{H}}_\nu$.

Similarly one constructs an isometric mapping F of the space $\mathfrak{H}_{\nu_1} \times \mathfrak{H}_{\nu_2}$ onto the space $\widetilde{\mathfrak{H}}_{\nu_1} \times \widetilde{\mathfrak{H}}_{\nu_2}$ of all measurable functions $\widetilde{\varphi}(w_1, w_2)$ for which

$$\int |w_1|^{-\nu_1} |w_2|^{-\nu_2} |\widetilde{\varphi}(w_1, w_2)|^2\, dw_1\, dw_2 < \infty$$

with scalar product

$$(\widetilde{\varphi}_1, \widetilde{\varphi}_2) = a(\nu_1) a(\nu_2) \int |w_1|^{-\nu_1} |w_2|^{-\nu_2} \widetilde{\varphi}_1(w_1, w_2) \overline{\widetilde{\varphi}_2(w_1, w_2)}\, dw_1\, dw_2.$$

The mapping F is the extension by continuity of the Fourier transform of functions $\varphi(z_1, z_2) \in K(Z \times Z)$, where $K(Z \times Z)$ is the totality of all finite infinitely differentiable functions of z_1, z_2.

The space $\mathfrak{H}_{\nu_1} \times \mathfrak{H}_{\nu_2}$ consists of precisely those generalized functions on $K(Z \times Z)$ whose Fourier transforms are ordinary functions belonging to $\widetilde{\mathfrak{H}}_{\nu_1} \times \widetilde{\mathfrak{H}}_{\nu_2}$.

Theorem. If $\nu_1 + \nu_2 \leqslant 2$, then $\mathfrak{D}_{\nu_1} \times \mathfrak{D}_{\nu_2}$ is unitarily equivalent to $\mathfrak{S}_{m_1\sigma_1} \times \mathfrak{S}_{m_2\sigma_2}$ for even $m_1 + m_2$ (in particular, it is unitarily equivalent to $\mathfrak{S}_{00} \times \mathfrak{S}_{00}$) and therefore there is a continuous sum of representations $\mathfrak{S}_{m\sigma}$ with even m; if, however, $\nu_1 + \nu_2 > 2$, then $\mathfrak{D}_{\nu_1} \times \mathfrak{D}_{\nu_2}$ is unitarily equivalent to the direct sum of the representation $\mathfrak{D}_{\nu_1+\nu_2-2}$ and $\mathfrak{S}_{m_1\sigma_1} \times \mathfrak{S}_{m_2\sigma_2}$ for even $m_1 + m_2$, and therefore there is a direct sum of the representation $\mathfrak{D}_{\nu_1+\nu_2-2}$ and a continuous sum of representations $\mathfrak{S}_{m\sigma}$ with even m.

We outline the proof of this theorem. First consider the case $\nu_1 = \nu_2$. Put $\nu = \nu_1 = \nu_2$ and define an operator A from $\mathfrak{H}_\nu \times \mathfrak{H}_\nu$ into $L^2(Z \times Z)$ with domain

$$D = \{f : f \in \mathfrak{H}'_\nu \times \mathfrak{H}'_\nu,\ |z_1 - z_2|^\nu f \in L^2(Z \times Z)\},$$

by setting, for $f \in D$, $Af = f'$, where

$$f'(z_1, z_2) = |z_1 - z_2|^\nu f(z_1, z_2);$$

D is dense in $\mathfrak{H}_\nu \times \mathfrak{H}_\nu$, since $D \supset K(Z \times Z)$. It is easy to verify that D is invariant with respect to the operators T_g of the representation $\mathfrak{D}_\nu \times \mathfrak{D}_\nu$, and that for $f \in D$

$$AT_g f = T'_g Af, \tag{2}$$

where $g \mapsto T'_g$ is the representation $\mathfrak{S}_{00} \times \mathfrak{S}_{00}$. It can be shown that:

1) A^* is defined on a set D^* dense in $L^2(Z)$ (namely, D^* contains the set K_0 of all functions $\varphi \in K(Z \times Z)$ whose support does not contain the diagonal $z_1 = z_2$), and therefore A admits a closure.

2) The orthogonal complement $R^{*\perp}$ of the range R^* of the operator A^* consists of precisely those generalized functions $f \in \mathfrak{H}_\nu \times \mathfrak{H}_\nu$ which vanish on K_0, hence which are concentrated on the diagonal $z_1 = z_2$ and therefore have the form

$$f(z_1, z_2) = \sum_{kj} \varphi_{kj}(z_2) D_j^{(k)} \delta(z_1 - z_2),$$

where $D_j^{(k)}$ denote derivatives of k-th order with respect to $x = \operatorname{Re} z$, $y = \operatorname{Im} z$ (see (1), Chapter II, §§ 4 and 5). Hence, from the condition $f \in \mathfrak{H}_\nu \times \mathfrak{H}_\nu$ (therefore,

$\tilde{f} \in \tilde{\mathfrak{H}}_\nu \times \tilde{\mathfrak{H}}_\nu$ it follows that, for $\nu \leq 1$, $R^{*\perp} = (0)$, and therefore R^* is dense in $\mathfrak{H}_\nu \times \mathfrak{H}_\nu$, while for $\nu > 1$, $R^{*\perp}$ consists of*

$$f(z_1, z_2) = \delta(z_1 - z_2)\psi(z_2), \qquad \psi(z_2) \in \mathfrak{H}_{2\nu-2}. \tag{3}$$

It follows from (3) that, for $\nu > 1$: a) $R^{*\perp}$ is invariant with respect to the operators of the representation $\mathfrak{D}_\nu \times \mathfrak{D}_\nu$; b) the correspondence $f \to \psi$, established by formula (3), defines an isometric mapping of $R^{*\perp}$ onto $\mathfrak{H}_{2\nu-2}$, under which the restriction of $\mathfrak{D}_\nu \times \mathfrak{D}_\nu$ to $R^{*\perp}$ passes into $\mathfrak{D}_{2\nu-2}$. From these listed properties of the operator A and the generalized Schur lemma (see ([2]), theorem 1, item 2, § 21) we conclude that the assertion of the theorem is valid for $\nu_1 = \nu_2$.

Let now $\nu_1 \neq \nu_2$, and, for definiteness, let $\nu_1 < \nu_2$; put $\nu = \nu_2 - \nu_1$. Define an operator A from $\mathfrak{H}_{\nu_1} \times \mathfrak{H}_{\nu_2}$ into $L^2(Z) \times \mathfrak{H}_\nu$ with domain of definition

$$D = \{f;\ f \in \mathfrak{H}'_{\nu_1} \times \mathfrak{H}'_{\nu_2};\ |z_1 - z_2|^{\nu_1} f \in L^2(Z) \times \mathfrak{H}'_\nu\},$$

putting, for $f \in D$, $Af = f'$, where $f'(z_1, z_2) = |z_1 - z_2|^{\nu_1} f(z_1, z_2)$. Again D is invariant with respect to the operators T_g, and again (2) is satisfied, where now $g \to T_g$ is the representation $\mathfrak{S}_{00} \times \mathfrak{D}_\nu$. The operator A^* is defined on a dense set D^*, and for the orthogonal complement $R^{*\perp}$ of the range R^* of the operator A we have:

1′) $R^{*\perp} = (0)$, and therefore R^* is dense in $\mathfrak{H}_{\nu_1} \times \mathfrak{H}_{\nu_2}$ for $\nu_1 + \nu_2 \leq 2$;

2′) $R^{*\perp}$ consists of all functions

$$f(z_1, z_2) = \delta(z_1 - z_2)\psi(z_2), \qquad \psi(z_2) \in \mathfrak{H}_{\nu_1+\nu_2-2} \tag{4}$$

for $\nu_1 + \nu_2 > 2$.

In case 2′): a') $R^{*\perp}$ is invariant with respect to the operators of the representation $\mathfrak{D}_{\nu_1} \times \mathfrak{D}_{\nu_2}$; b') the correspondence $f \to \psi$, established by formula (4), is an isometric mapping of $R^{*\perp}$ onto $\mathfrak{H}_{\nu_1+\nu_2-2}$, under which the restriction of $\mathfrak{D}_{\nu_1} \times \mathfrak{D}_{\nu_2}$ to $R^{*\perp}$ passes into $\mathfrak{D}_{\nu_1+\nu_2-2}$. Therefore, applying the generalized Schur lemma, we conclude that, for $\nu_1 + \nu_2 \leq 2$, the representation $\mathfrak{D}_{\nu_1} \times \mathfrak{D}_{\nu_2}$ is unitarily equivalent to the representation $\mathfrak{S}_{00} \times \mathfrak{D}_\nu$, while for $\nu_1 + \nu_2 > 2$ it is equivalent to the direct sum of the representations $\mathfrak{S}_{00} \times \mathfrak{D}_\nu$ and $\mathfrak{D}_{\nu_1+\nu_2-2}$.

But, on the other hand, by virtue of the principal result in ([5]), the representation $\mathfrak{S}_{00} \times \mathfrak{D}_\nu$ is unitarily equivalent to the representation $\mathfrak{S}_{00} \times \mathfrak{S}_{00}$. Hence the assertion of the theorem follows for $\nu_1 \neq \nu_2$.

Remark 1. Let $\nu_1 = \nu_2 = \nu$.

Denote by $\mathcal{H}_\nu$ the Hilbert space of all measurable functions $f(z_1, z_2)$ for which

$$\int |z_1 - z_2|^{2\nu} |f(z_1, z_2)|^2 \, dz_1 \, dz_2 < \infty$$

with scalar product

$$(f_1, f_2)_1 = \int |z_1 - z_2|^{2\nu} f_1(z_1, z_2)\overline{f_2(z_1, z_2)}\, dz_1\, dz_2.$$

Define in $\mathcal{H}_\nu$ the representation $\mathfrak{E}_\nu$ by the same formula (1); it is easy to verify that $\mathfrak{E}_\nu$ is a unitary representation of the group $\mathfrak{G}$.

The correspondence $f(z_1, z_2) \to |z_1 - z_2|^\nu f(z_1, z_2)$ is an isometric mapping of $\mathcal{H}_\nu$ onto $L^2(Z)$, carrying $\mathfrak{E}_\nu$ into $\mathfrak{S}_{00} \times \mathfrak{S}_{00}$; therefore, from the theorem proved, we conclude that, for $\nu \leq 1$, the representation $\mathfrak{D}_\nu \times \mathfrak{D}_\nu$ is unitarily equivalent to $\mathfrak{E}_\nu$, while for $\nu > 1$ the representation $\mathfrak{D}_\nu \times \mathfrak{D}_\nu$ is unitarily equivalent to the direct sum of the representations $\mathfrak{E}_\nu$ and $\mathfrak{D}_{2\nu-2}$.

Let now $\nu_1 \neq \nu_2$, and, for definiteness, let $\nu_1 < \nu_2$. Denote by $\mathcal{H}'_{\nu_1\nu_2}$ the totality of all measurable functions $f(z_1, z_2)$ for which

$$\int |z_1 - z_1'|^{-2+\nu_2-\nu_1} |z_1 - z_2|^{\nu_1} |z_1' - z_2'|^{\nu_1} |f(z_1, z_2)|\, |f(z_1', z_2')|\, dz_1\, dz_2\, dz_2' < \infty$$

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* (3) means that, for $\varphi \in K(Z \times Z)$,

$$(f; \varphi(z_1, z_2)) = (\psi; \varphi(z, z)).$$

with the scalar product

$$(f_1, f_2)_2 = \int |z_1 - z_1'|^{-2+\nu_2-\nu_1} |z_1' - z_2|^{\nu_1} |z_1 - z_2'|^{\nu_1} f_1(z_1, z_2) f_2(z_1, z_2')\, dz_1\, dz_2\, dz_2',$$

and by $\mathcal{H}'_{\nu_1\nu_2}$ the completion of $\mathcal{H}_{\nu_1\nu_2}$ with respect to the scalar product $(f_1, f_2)_2$. Define in $\mathcal{H}_{\nu_1\nu_2}$ the representation $\mathfrak{S}_{\nu_1\nu_2}$ by the same formula (1); it is easy to verify that $\mathfrak{S}_{\nu_1\nu_2}$ is a unitary representation of the group $\mathfrak{G}$. The correspondence $f(z_1, z_2) \to |z_1 - z_2|^{\nu_1} f(z_1, z_2)$ is an isometric mapping of $\mathcal{H}_{\nu_1\nu_2}$ onto $L^2(Z) \times \mathfrak{H}_{\nu_2-\nu_1}$, carrying $\mathfrak{S}_{\nu_1\nu_2}$ into $\mathfrak{S}_{00} \times \mathfrak{D}_{\nu_2-\nu_1}$; therefore, from the theorem proved above we conclude that for $\nu_1 + \nu_2 \leq 2$ the representation $\mathfrak{D}_{\nu_1} \times \mathfrak{D}_{\nu_2}$ is unitarily equivalent to $\mathfrak{S}_{\nu_1\nu_2}$, while for $\nu_1 + \nu_2 > 2$ the representation $\mathfrak{D}_{\nu_1} \times \mathfrak{D}_{\nu_2}$ is unitarily equivalent to the direct sum of the representations $\mathfrak{S}_{\nu_1\nu_2}$ and $\mathfrak{D}_{\nu_1+\nu_2-2}$.

Remark 2. The fact that for $\nu_1 + \nu_2 > 2$ the representation $\mathfrak{D}_{\nu_1} \times \mathfrak{D}_{\nu_2}$ must contain $\mathfrak{D}_{\nu_1+\nu_2-2}$ was noted by I. M. Gel' fand, on the basis of the following consideration.

Let the representation $\mathfrak{D}_{\nu_1} \times \mathfrak{D}_{\nu_2}$ decompose into a direct sum of representations $\mathfrak{S}_{m\sigma}$, and let $f_1 \in \mathfrak{H}_{\nu_1}$, $f_2 \in \mathfrak{H}_{\nu_2}$ be normalized elements invariant with respect to the restrictions of $\mathfrak{D}_{\nu_1}$, $\mathfrak{D}_{\nu_2}$ to the unitary subgroup. Then $(T_\varepsilon(f_1 \times f_2), f_1 \times f_2)$ is equal to the product of the spherical functions of the representations $\mathfrak{D}_{\nu_1}$, $\mathfrak{D}_{\nu_2}$, i.e.

$$(T_\varepsilon(f_1 \times f_2), f_1 \times f_2) = \frac{4}{\nu_1\nu_2} \frac{\operatorname{sh}\nu_1 t \cdot \operatorname{sh}\nu_2 t}{\operatorname{sh}^2 2t} \quad \text{for } \varepsilon = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix}.$$

On the other hand, applying to $\mathfrak{D}_{\nu_1} \times \mathfrak{D}_{\nu_2}$ the decomposition into the representations $\mathfrak{S}_{m\sigma}$, we obtain that

$$\frac{4 \operatorname{sh}\nu_1 t \ \operatorname{sh}\nu_2 t}{\nu_1\nu_2 \operatorname{sh}^2 2t} = (T_\varepsilon(f_1 \times f_2), f_1 \times f_2) = \int_0^\infty \frac{2}{\rho} \frac{\sin\rho t}{\operatorname{sh} 2t}\, d\mu(\rho),$$

where μ is some nondecreasing function on $[0, \infty)$ and

$$\int_0^\infty d\mu(\rho) = 1,$$

while $\dfrac{2}{\rho}\dfrac{\sin\rho t}{\operatorname{sh} 2t}$ is the spherical function of the representation $\mathfrak{S}_{0\rho}$. Hence

$$\frac{4 \operatorname{sh}\nu_1 t \ \operatorname{sh}\nu_2 t}{\nu_1\nu_2 \operatorname{sh} 2t} = \int_0^\infty \frac{2}{\rho} \sin\rho t\, d\mu(\rho), \tag{5}$$

but this is impossible for $\nu_1 + \nu_2 > 2$, since then the left-hand side is unbounded as $t \to \infty$, whereas the right-hand side is $\leq 2/\rho$. If, however, from

$$\frac{4 \operatorname{sh}\nu_1 t \ \operatorname{sh}\nu_2 t}{\nu_1\nu_2 \operatorname{sh}^2 2t}$$

one subtracts the product by

$$2(\nu_1 + \nu_2 - 2)\nu_1^{-1}\nu_2^{-1}$$

of the spherical function

$$\frac{2}{\nu_1 + \nu_2 - 2} \frac{\operatorname{sh}(\nu_1 + \nu_2 - 2)t}{\operatorname{sh} 2t}$$

of the representation $\mathfrak{D}_{\nu_1+\nu_2-2}$, then the product by $\operatorname{sh} 2t$ of the resulting difference will already be bounded, and for it a representation in the form of the integral on the right-hand side of (5) is already possible.

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Note: Figure translations are in progress. See original paper for figures.