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Abstract

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MATHEMATICS

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PERIODIC MOTIONS OF A VISCOUS INCOMPRESSIBLE FLUID

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A theorem is proved on the existence "in the large" of a periodic solution of a nonlinear ordinary differential equation in Hilbert space. The result obtained is then applied to the study of periodic regimes of motion of a viscous incompressible fluid. We note that the corresponding initial-value problem was studied in ($^{1-6}$).

1. Consider in a separable Hilbert space H the ordinary differential equation of first order

$$Lx \equiv dx/dt + Ax + Kx = f. \tag{1,1}$$

We make the following assumptions: 1) $x = x(t)$, $f = f(t) \in H$ for fixed $t \in (-\infty, +\infty)$; 2) A is a time-independent linear operator with domain of definition D_A dense in H (the elements of D_A will henceforth be called smooth); A is symmetric, positive definite: for any smooth Φ, $(A\Phi, \Phi)_H \geq \gamma^2 \|\Phi\|_H^2$; A^{-1} is completely continuous; 3) Kx is a time-independent nonlinear operator acting from D_A into H and such that, for smooth Φ, $(K\Phi, \Phi)_H \geq 0$; 4) the operator $A^{-2}Kx$ can be extended to the whole space H as a continuous one; 5) $f(t)$ is a T-periodic function of t: $f(t+T) \equiv f(t)$; 6) $\max_t \|f\|_H < \infty$.

By a generalized T-periodic solution of equation (1,1) we mean a vector-function $x(t)$ satisfying the conditions:

a) $x(t+T) \equiv x(t)$; b) $\max_t \|x\|_H < \infty$; $\int_0^t \|A^{1/2}x\|_H^2 \, dt < \infty$; c) for every $\Phi(t) \in D_{A^2}$, strongly differentiable with respect to t and T-periodic, the identity holds

$$\int_0^T \left[\left(-x, \frac{d\Phi}{dt}\right)_H + \left(A^{1/2}x, A^{1/2}\Phi\right)_H + \left(A^{-2}Kx, A^2\Phi\right)_H \right] dt = \int_0^T (f, \Phi)_H \, dt. \tag{1,2}$$

Theorem 1. *Let conditions 1)–6) be satisfied. Then equation (1,1) has at least one generalized T-periodic solution.*

By imposing smoothness conditions on K, one can obtain a uniqueness theorem for periodic motion for small f. It is sufficient, for example, to require the condition

$$\left\|A^{-1/2}Kx_1 - A^{-1/2}Kx_2\right\| \leq \varphi(\|x_1\|, \|x_2\|) \left\|A^{1/2}x_1 - A^{1/2}x_2\right\|,$$

where $\varphi(x_1, x_2) \to 0$, when $\|x_1\|, \|x_2\| \to 0$.

For the proof we use the Galerkin method in the following form. Construct a sequence $\{x_n\}$ of the form

$$x_n = \sum_{k=1}^{n} A_{kn}(t)\varphi_k,$$

where $\{\varphi_k\}$ is some system complete in H, with $\varphi_k \in D_A$. The vector $(A_{1n}(t); A_{2n}(t), \ldots$

$\ldots, A_{nn}(t))$ is determined as a periodic solution of the nonlinear system of ordinary differential equations

$$(Lx_n, \varphi_k)_H = (f, \varphi_k)_H \quad (k = 1, 2, \ldots, n). \tag{1,3}$$

Let us prove that system (1,3) admits at least one T-periodic solution for every n. Multiplying the k-th equation of system (1,3) by A_{kn} and summing over k from 1 to n, we obtain the energy equation in the form

$$\frac{1}{2}\frac{d}{dt}\|x_n\|_H^2 + \|A^{1/2}x_n\|_H^2 \leq (f, x_n)_H. \tag{1,4}$$

Using equation (1,4), it is easy to prove that system (1,3), for arbitrary initial data $(A_{10}, A_{20}, \ldots, A_{n0})$, has a solution $(A_1(t), A_2(t), \ldots, A_n(t))$, bounded for $0 \leq t \leq \infty$. Thus one can define, in the n-dimensional space of coefficients $(A_1, \ldots, A_n)$, an operator K by setting

$$K(A_{10}, A_{20}, \ldots, A_{n0}) = (A_1(T), A_2(T), \ldots, A_n(T)).$$

The transformation introduced here is continuous. Using equation (1,4), one can establish that the sphere $\|x_n\| \leq R$, for sufficiently large R, is mapped by the operator K into itself. By Brouwer' s theorem (7) we conclude that there exists at least one fixed point. The corresponding solution of system (1,3) will, obviously, be periodic.

Furthermore, it is not difficult to show that the following estimates are valid for x_n:

$$\|x_n\| \leq \frac{1}{\gamma^2} \max_t \|f\|, \qquad \int_0^T \|A^{1/2} x_n\|^2 \, dt \leq \frac{1}{\gamma^2} \int_0^T \|f\|^2 \, dt. \tag{1,5}$$

The existence of a generalized periodic solution of equation (1,1) is now easily established by passing to $n \to \infty$. Imposing further restrictions on the operator K, one can carry out an investigation of the smoothness of the solutions obtained. Finally, note that one can consider an equation somewhat more general than (1,1), adding to the left-hand side of (1,1) a small operator $K_1 x$ and allowing A to depend on t T-periodically.

2. Let a domain Ω of two- or three-dimensional space be filled with a viscous incompressible fluid acted upon by vortex body forces $\mathbf{f}$ periodic in time with period T. We regard the velocity on the boundary S of the domain Ω as known and also T-periodic in time. We pose the problem of finding T-periodic flows of the fluid: to find a vector $\mathbf{v}(x,t)$ and a function $p(x,t)$ $(x \in \Omega;\ t \in (-\infty, +\infty))$ under the conditions

$$L\mathbf{v} \equiv \partial \mathbf{v}/\partial t + (\mathbf{v}, \nabla)\mathbf{v} - \nu \Delta \mathbf{v} = \mathbf{f} - \frac{1}{\rho} \nabla p, \qquad \nu, \rho = \text{const} > 0; \tag{2,1}$$

$$\operatorname{div} \mathbf{v} = 0; \tag{2,2}$$

$$\mathbf{v}\big|_S = \vec{\alpha}; \tag{2,3}$$

$$\mathbf{v}(x, t+T) \equiv \mathbf{v}(x,t), \qquad p(x, t+T) \equiv p(x,t). \tag{2,4}$$

Suppose the following conditions are satisfied: 1) Ω is a bounded domain with boundary consisting of m closed surfaces $S_1, S_2, \dots, S_m$ with continuous curvature*; 2) there exists a vector $\mathbf{a}(x,t)$, continuously differentiable in t and twice continuously differentiable in x_i, such that

$$\operatorname{div} \mathbf{a} = 0; \qquad \mathbf{a}\big|_S = \vec{\alpha}; \qquad \mathbf{a}(x, t+T) \equiv \mathbf{a}(x,t);$$

3) $\vec{\alpha}$ satisfies the conditions

$$\int_{S_k} \vec{\alpha} \cdot \mathbf{n} \, dS = 0 \quad (k = 1, \dots, m);$$

$\mathbf{n}$ is the normal to S. This condition physically means the absence of sources inside the surfaces S_k. In the case $m = 1$ it is a consequence of (2,2).

* If $\vec{\alpha}\big|_S = 0,$ the smoothness condition on the boundary is superfluous.

Introduce the following function spaces:

H_1' is the closure of the set Q of smooth T-periodic solenoidal vectors vanishing near S, in the norm generated by the scalar product

$$(\mathbf{u}, \mathbf{w})_{H_1'} = \int_0^T \int_\Omega \operatorname{rot} \mathbf{u} \cdot \operatorname{rot} \mathbf{w} \, dx \, dt. \tag{2,5}$$

H_r is the closure of the set Q in the norm

$$\|\mathbf{u}\|_{H_1}^r = \int_0^T \int_\Omega \left(\left| \frac{\partial \mathbf{u}}{\partial t} \right|^r + |\Delta \mathbf{u}|^r \right) dx \, dt. \tag{2,6}$$

A generalized T-periodic solution of problem (2,1)–(2,4) is a vector $\mathbf{v} = \mathbf{a} + \mathbf{u}$, where $\mathbf{u} \in H_1'$, $\max_t \|\mathbf{u}\|_{L_2(\Omega)} < \infty$, and, for every T-periodic $\vec{\Phi}(x,t) \in S$, satisfies the integral identity

$$\int_0^T \int_\Omega \left[\frac{\partial \vec{\Phi}}{\partial t} \mathbf{u} - (\mathbf{u}, \nu)\mathbf{u} \cdot \vec{\Phi} - (\mathbf{u}, \nu)\mathbf{a} \cdot \vec{\Phi} - (\mathbf{a}, \nu)\mathbf{u} \cdot \vec{\Phi} - \nu \operatorname{rot} \mathbf{u} \cdot \operatorname{rot} \vec{\Phi} + \mathbf{f}_1 \cdot \vec{\Phi} \right] dx \, dt = 0, \tag{2,7}$$

where $\mathbf{f}_1 = \mathbf{f} - \partial \mathbf{a}/\partial t - (\mathbf{a}, \nu)\mathbf{a} + \nu \Delta \mathbf{a}$.

To reduce the problem of finding $\mathbf{u}(x,t)$ to an ordinary differential equation in the space $L_2(\Omega)$, it suffices, as was done in ([6],[8]), to apply to equation (2,1) the operator P of orthogonal projection in $L_2(\Omega)$ onto the subspace H', the closure of the set of smooth solenoidal vectors equal to 0 near S. With the aid of the results of item 1, Theorem 2 is derived.

Theorem 2. Suppose that conditions 1)–4) and, in addition, $\max_t \|\mathbf{f}\|_{L_2(\Omega)} < \infty$ are satisfied. Then problem (2,1)–(2,4) has at least one generalized T-periodic solution.

For the study of the differential properties of the generalized solution obtained, the corresponding linear problem is first considered.

Theorem 3*. Let $\mathbf{u}(x,t)$ be a generalized T-periodic solution** of the problem

$$\partial \mathbf{u}/\partial t - \nu \Delta \mathbf{u} = \mathbf{F}(x,t) - \frac{1}{\rho} \nabla p,$$

$$\operatorname{div} \mathbf{u} = 0, \qquad \mathbf{u}|_S = 0, \qquad \mathbf{u}(x, t+T) \equiv \mathbf{u}(x,t), \tag{2,8}$$

Assume that S has continuous curvature, $\mathbf{F} \in L_r$ $(r > 1)$ in $Q_T = \Omega \times [0, T]$, and $\mathbf{F}(x, t + T) \equiv \mathbf{F}(x, t)$. Then there exist generalized derivatives $\partial \mathbf{u}/\partial t$, $\partial^2 \mathbf{u}/\partial x_i \partial x_k$, and

$$\|\mathbf{u}\|_{H_r} \leq C \left(\|\mathbf{F}\|_{L_r(Q_T)} + \max_t \|\mathbf{u}\|_{L_2(\Omega)} \right). \tag{2,9}$$

If $\mathbf{F} \in W_r^{(l)}$ in Q_T and the boundary S is $l + 2$ times continuously differentiable, then

$$\int_0^T \int_\Omega \left(\left| \frac{\partial}{\partial t} D^l \mathbf{u} \right|^r + \sum_{i,k} \left| \frac{\partial^2}{\partial x_i \partial x_k} D^l \mathbf{u} \right|^r \right) dx\, dt \leq C \left(\|\mathbf{F}\|_{W_r^{(l)}(Q_T)} + \max_t \|\mathbf{u}\|_{L_2(\Omega)} \right); \tag{2,10}$$

D^l is any derivative of order l with respect to x_i, t.

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* The corresponding theorems were also obtained by us for the problem with initial data.

** Introduced by equality (2,9), in which $\mathbf{a} = 0$, $\mathbf{f} = \mathbf{F}$, and the nonlinear term is discarded.

Using Theorem 3 and an inequality of O. A. Ladyzhenskaya(4), one derives

Theorem 4. *Suppose the conditions of Theorem 2 are fulfilled and* $\mathbf{f}(x, t) \in L_r(Q_T)$ $(r \geq 2)$. *Then the generalized solution of problem (2.1)–(2.4)* $\mathbf{v} = \mathbf{a} + \mathbf{u}$, *where* $\mathbf{u} \in H_{5/4}$ *in the 3-dimensional case,* $\mathbf{u} \in H_r$ *in the 2-dimensional case.*

Furthermore, in the 2-dimensional case, if $\mathbf{f} \in W_r^{(l)}(Q_T)$, *then* $\frac{\partial}{\partial t} D^l \mathbf{u}$, $\frac{\partial^2}{\partial x_i \partial x_k} D^l \mathbf{u} \in L_r(Q_T)$. *For* $l = 1$, $r > 2$, *the generalized solution will also be classical.*

The case of an unbounded domain Ω is exhausted by similar considerations. In the general case, apparently, more than one generalized periodic solution exists. For sufficiently small smooth $\mathbf{f}$ and $\mathbf{a}$, one can prove a uniqueness theorem for the solution of problem (2.1)–(2.4). "In the small" one can also carry out a further study of the differential properties of the solution of the 3-dimensional problem.

3. The Galerkin method in the form given in § 1 is often ineffective, since it is connected with finding periodic solutions of nonlinear systems of ordinary differential equations. Therefore it makes sense to consider another version of the Galerkin method, in which the matter reduces to solving algebraic systems. Let $\{\vec{\psi}_k(x)\}$ be a complete system in the space $H_{1\Omega}$* and let $\vec{\psi}_k(x) \in W_2^{(2)}$ in Ω. We shall seek an approximate solution of problem (2.1)–(2.4) in the form

$$\mathbf{v}_n = \mathbf{a} + \mathbf{u}_n = \mathbf{a} + \sum_{k,l=0}^{n} \left[A_{kl} \vec{\psi}_l(x) \sin kt + B_{kl} \vec{\psi}_l(x) \cos kt \right]; \qquad A_{0l} = 0. \quad (3.1)$$

A_{kl}, B_{kl} are constants determined from the algebraic system

$$\int_0^T \int_\Omega L\mathbf{v}_n \cdot \vec{\psi}_l(x) \sin kt \, dx \, dt = \int_0^T \int_\Omega \mathbf{f} \cdot \vec{\psi}_l(x) \sin kt \, dx \, dt,$$

$$\int_0^T \int_\Omega L\mathbf{v}_n \cdot \vec{\psi}_l(x) \cos kt \, dx \, dt = \int_0^T \int_\Omega \mathbf{f} \cdot \vec{\psi}_l(x) \cos kt \, dx \, dt \qquad (3.2)$$

$$(k = 0, 1, \dots, n; \; l = 0, 1, \dots, n).$$

The system (3.2) turns out to be solvable for every n. It is shown that the set $\{\mathbf{u}\}$ contains a sequence converging strongly in $L_{10/3}$ in the spatial case and in H_r in the planar case to the generalized solution of problem (2.1)–(2.4). If $\{\vec{\psi}_k(x)\}$ are chosen as in $^{(9)}$, one can obtain estimates of the rate of convergence in various norms.

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* The closure of the set of smooth solenoidal vectors equal to 0 near S in the norm generated by the scalar product

$$(\mathbf{u}, \mathbf{w})_{H_{1\Omega}} = \int_{\Omega} \operatorname{rot} \mathbf{u} \cdot \operatorname{rot} \mathbf{w} \, dx.$$

Note: Figure translations are in progress. See original paper for figures.

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