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1960

Abstract

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MATHEMATICS

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ON THE SINGULARITIES OF A CERTAIN CLASS OF DIRICHLET SERIES

(Presented by Academician I. M. Vinogradov on 7 XII 1959)

We shall consider Dirichlet series of the form

$$f(s) = \sum_{n=0}^{\infty} a_n e^{-\lambda_n s}, \quad 0 \leq \lambda_0 < \lambda_1 < \lambda_2 < \cdots \quad (s = \sigma + i\tau);$$

$$\lim(\lambda_{n+1} - \lambda_n) \geq h > 0. \tag{1}$$

We shall denote by D_λ the maximum density of the sequence $\{\lambda_n\}$ in the sense of Pólya. It is known that the series (1) preserves, in a weaker form, certain properties inherent in the series

$$\sum_{n=0}^{\infty} be^{-ns}. \tag{2}$$

Thus, every segment of the axis of convergence of the series (1) of length greater than $2\pi D_\lambda$ contains at least one singular point of $f(s)$ (Pólya' s theorem).

If a segment of the axis of convergence of the series (1) of length greater than $2\pi(D_\lambda + h^{-1})$ contains only simple poles of $f(s)$, $\sigma_c + i\alpha_1$, $\sigma_c + i\alpha_2, \dots, \sigma_c + i\alpha_k$, then every isolated singular point $\sigma_c + i\alpha$ of $f(s)$ on the axis of convergence is a simple pole of $f(s)$, and

$$\alpha = m_1\alpha_1 + m_2\alpha_2 + \cdots + m_k\alpha_k, \quad \text{where } m_j \text{ are integers} \tag{3}$$

(Agmon' s theorem (3)).

We shall show that certain additional conditions can ensure a distribution of the poles of $f(s)$ on the axis of convergence close to periodic. We shall apply the results obtained to the study of solutions of Riemann' s functional equation (Theorem 4).

For simplicity we shall assume that the abscissa of convergence of the series (1) is $\sigma_c = 0$. Put $\delta_m = \lambda_{m+1} - \lambda_m$; $((x))$ will denote the distance from x to the nearest integer.

We shall say that the sequence $\{\lambda_n\}$ contains a "regular part" of dimension r, if one can find an increasing sequence of indices $\{m_k\}$ and positive numbers $h_1, h_2, \dots, h_r$ such that

$$\delta_{m_k+lr+\nu} \to h_\nu \quad (m_k \to \infty), \quad \nu = 1, 2, \dots, r, \quad l = 0, 1, 2, \dots . \tag{4}$$

We shall say that the sequence $\{\lambda_n\}$ contains an "almost regular part" of dimension r, if there exists an increasing sequence of indices $\{n_k\}$ and positive numbers $g_1, g_2, \dots, g_r$ such that

$$\left(\left(g_\nu^{-1}\delta_{n_k+lr+\nu}\right)\right) \to 0 \quad \text{as } n_k \to \infty, \quad \nu = 1, 2, \dots, r, \quad l = 0, 1, 2, \dots . \tag{5}$$

Lemma. If $0 < h < \delta_m < H$ for all $m > m_0$ and the sequence $\{\lambda_n\}$ contains an "almost regular part," then it also contains a "regular part."

Proof. Since $\delta_m < H$ for $m > m_0$, for each ν the integer nearest to the number $g_\nu^{-1}\delta_{n_k+lr+\nu}$ can take only a finite number of values. Therefore one can extract from $\{n_k\}$ a sequence $\{m_k\}$ such that

$$\delta_{m_k+lr+\nu} \to p_\nu g_\nu$$

as $m_k \to \infty$, $\nu = 1, 2, \dots, r$, $l = 0, 1, 2, \dots$, where p_ν are integers. Since $\delta_m > h > 0$ for $m > m_0$, we have $p_\nu > 0$, $\nu = 1, 2, \dots, r$. Thus the lemma is proved, with $h_\nu = p_\nu g_\nu$.

Theorem 1. Let the sequence of exponents $\{\lambda_n\}$ of the series (1) contain an "almost regular part" of dimension 1. Let the only singularities of $f(s)$ on a segment L of the imaginary axis of length $|L| > 2\pi h^{-1}$ be simple poles at the points $i\alpha_1, i\alpha_2, \dots, i\alpha_k$. Let $i\alpha$ be any simple pole of $f(s)$ on the imaginary axis. Then the equality

$$\alpha = \alpha_q + 2\pi d N_\alpha, \tag{6}$$

holds, where N_α is an integer, d is a positive number, and α_q is one of the numbers $\alpha_1, \alpha_2, \dots, \alpha_k$ specified in the hypothesis of the theorem. Moreover,

$$\operatorname{Res}_f(i\alpha) = \operatorname{Res}_f(i\alpha_q) \cdot \lim \exp\left[i(\alpha - \alpha_q)m_k''\right] \quad (m_k'' \to \infty), \tag{7}$$

where $\{m_k''\}$ is some sequence, one and the same for all α.

Proof. From the assumption on the singularities of $f(s)$ on L it follows that the sequence of coefficients $\{a_n\}$ is almost regular, i.e.

$$0 < C_1 < \max_{\nu=1,2,\dots,N} |a_{n+\nu}| < C_2$$

for some N and all sufficiently large n, and also that

$$\lambda_{n+1} - \lambda_n < H$$

for sufficiently large n ([1–3]).

In this case, as Agmon showed ([1,3]), the family of functions

$$f_{n_k}(s) = \left[f(s) - \sum_{n<n_k} a_n \exp(-\lambda_n s) \right] \exp(\lambda_{n_k} s)$$

(where $\{n_k\}$ is any sequence of indices) is uniformly bounded in every closed bounded domain contained in the domain of regularity of $f(s)$, and every function $g(s)$ that is a limit function for some subsequence $\{f_{n'_k}(s)\}$ of the functions of the family has the following properties: a) $g(s)$ is holomorphic and single-valued in the entire s-plane from which the singular points of $f(s)$ on the imaginary axis have been removed; b) every simple pole of $f(s)$ is a simple pole of $g(s)$, and

$$\mathrm{Res}_g(i\alpha) = \mathrm{Res}_f(i\alpha) \cdot \lim \exp(i\alpha n'_k) \quad (n'_k \to \infty); \tag{8}$$

c) for $\mathrm{Re}\, s > 0$

$$g(s) = \sum_0^\infty b_n \exp(-\mu_n s), \qquad |b_n| < c, \qquad 0 < h \le \mu_{n+1} - \mu_n \le H \tag{9}$$

(an analogous expansion holds for $\mathrm{Re}\, s < 0$), and one can extract from $\{n'_k\}$ a subsequence $\{n''_k\}$ such that

$$\mu_m = \lim(\lambda_{n''_k+m} - \lambda_{n''_k}) \quad (n''_k \to \infty), \qquad m = 0, 1, 2, \dots. \tag{10}$$

Let us now note that, since $0 < h < \delta_n < H$ for $n > n_0$, the conditions of the lemma are fulfilled, and therefore $\{\lambda_n\}$ contains a regular part.

Choose as $\{n_k\}$ the sequence $\{m_k\}$ mentioned in the definition of a regular part. Then (4) with $r = 1$ and (10) give $\mu_m = h_1 m$, and, in view of (9), $g(s)$ has period $2\pi h_1^{-1} i$ (or smaller). Since $h \le h_1$, it follows that $|L| > 2\pi h_1^{-1}$, and for every pole $i\alpha$ of $g(s)$ there is an "equivalent" one on the segment L, for example $i\alpha_q$, i.e.

$$\alpha = \alpha_q + 2\pi h_1^{-1} N_\alpha,$$

where

N_α is an integer. The residues of $g(s)$ at the points $i\alpha$ and $i\alpha_q$ coincide. But then (8) gives

$$\mathrm{res}_f(i\alpha) = \mathrm{res}_f(i\alpha_q) \cdot \lim \exp\{-i(\alpha - \alpha_q) m_k\} \qquad (m_k \to \infty),$$

and the proof of the theorem is complete.

Remark. If one assumes that $|L| > 2\pi(D_\lambda + h^{-1})$, then, by Armon' s theorem, every isolated singularity of $f(s)$ on the imaginary axis is a simple pole, and, consequently, formula (6) is valid for every isolated singularity of $f(s)$ on the imaginary axis.

Suppose now that all singularities of $f(s)$ on the imaginary axis are simple poles. Then from the proof of Theorem 1 it is clear that the sets of poles of $g(s)$ and $f(s)$ on the imaginary axis coincide, and therefore the poles of $f(s)$ are also periodically distributed. In this case, together with each pole at the point $i\alpha_q$, $f(s)$ and $g(s)$ have a pole at the point $i\alpha_q + i2\pi h_1^{-1}$; and therefore, by virtue of (7), the limit must exist

$$\lim_{m'_k\to\infty} \exp\{-(i\alpha_q + 2\pi i h_1^{-1} - i\alpha_q)m'_k\} = \\ = \lim_{m'_k\to\infty} \exp(-2\pi i h_1^{-1} m'_k) = \exp(-2\pi i\xi),$$

where one may assume that $0 \le \xi < 1$, and (7) gives, for $\alpha = \alpha_q + 2\pi h_1^{-1} N_\alpha$,

$$\mathrm{res}_f(i\alpha) = \mathrm{res}_f(i\alpha_q)\exp(-2\pi i N_\alpha \xi). \tag{11}$$

The function $\varphi(s) = \exp(h_1\xi s)f(s)$ then has periodically distributed poles on the imaginary axis, and the residue in each series of poles (i.e., of poles at the points $i\alpha_q + i2\pi h_1^{-1}N$, where α_q is fixed and N runs through all integer values) is one and the same. Thus the following has been proved:

Theorem 2. If, in addition to the hypotheses of Theorem 1, one assumes that $f(s)$ has as singularities on the imaginary axis only simple poles, then these poles are distributed periodically with period $2\pi i h_1^{-1}$ (or a smaller one), and the residues for each series of poles form a geometric progression with ratio $\exp(-2\pi i\xi)$, the same for all series. The function $\varphi(s) = \exp(h_1\xi s)f(s)$ has periodically distributed poles on the imaginary axis with residues equal within each series.

Theorem 3. Let the sequence of exponents of the series (1) contain an "almost regular part" of dimension r, and let all singularities of the sum of the series $f(s)$ on the imaginary axis (the axis of convergence) be simple poles. Then the set T of all poles of $f(s)$ on the imaginary axis can be represented as the sum of sets T_ν, $\nu = 1, 2, \ldots, r$, where in each set the poles are distributed periodically. One may specify a common period, independent of ν. The residues of each series of poles of each T_ν form a geometric progression with ratio $\exp(2\pi i\xi_\nu)$, $0 \le \xi_\nu < 1$, where ξ_ν depends on the set T_ν, but does not depend on the choice of the series within the set. Here it is assumed that if some poles of several different sets T_ν coincide, then their residues are added (so that at some points the poles may even cancel).

Proof is carried out essentially in the same way as in Theorems 1 and 2. Denote $h_1 + h_2 + \dots + h_\nu = c_\nu$, $\nu = 1, 2, \dots, r-1$, $h_1 + h_2 + \dots + h_r = c$. The limiting function $g(s)$ (corresponding to the "regular part" $\{\lambda_n\}$) can be represented, if one sums over progressions, in the form

$$g(s) = \sum_{n=0}^{\infty} a_{nr} \exp(-cns) + \exp(-c_1 s) \sum_{n=0}^{\infty} a_{nr+1} \exp(-cns) + \cdots$$

$$\cdots + \exp(-c_{r-1} s) \sum_{n=0}^{\infty} a_{nr+r-1} \exp(-cns)$$

or

$$g(s) = g_0(s) + \exp(-c_1 s) g_1(s) + \cdots + \exp(-c_{r-1} s) g_{r-1}(s). \tag{12}$$

Putting in (12) $s = s$, $s + 2\pi i c^{-1}$, $s + 4\pi i c^{-1}, \dots, s + 2(r-1)\pi i c^{-1}$, we obtain a system of equations with respect to $g_0(s)$ and $\exp(-c_\nu s) g_\nu(s)$ with a Vandermonde determinant different from zero. Consequently, $g_0(s)$ and $\exp(-c_\nu s) g_\nu(s)$ are expressed linearly in terms of $g(s + 2\pi i k c^{-1})$, $k = 0, 1, \dots, r-1$, and can have on the imaginary axis only simple poles, and the same is true for $g_\nu(s)$. But $g_0(s)$ and $g_\nu(s)$ ($\nu = 1, 2, \dots, r-1$) are periodic functions of s with period $2\pi c^{-1} i$. Therefore the poles of $g_0(s)$, $g_\nu(s)$ are distributed periodically on the imaginary axis, and the same is true for $\exp(-c_\nu s) g_\nu(s)$. The residues for each series of poles of any of the functions $g_0(s)$, $g_\nu(s)$, $\nu = 1, 2, \dots, r-1$, form a geometric progression with ratio, respectively, 1 or $\exp(-2\pi i c_\nu c^{-1})$. Finally, formula (8) and arguments analogous to those used in the proof of formula (11) complete the proof of Theorem 3.

Remark. Agmon' s theorems, as well as Theorems 1, 2, and 3 of the present paper, can be generalized to the case of multiple poles and somewhat more general singularities. One can also study series of the type

$$f(s) = \sum_{n=0}^{\infty} p_n(s) \exp(-\lambda_n s),$$

where $p_n(s)$ are polynomials whose degrees are bounded.

Theorem 4. Let

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{\mu_n^s} \quad \text{and} \quad g(s) = \sum_{n=1}^{\infty} \frac{b_n}{\lambda_n^s}$$

be Dirichlet series forming solutions (in the sense of Chandrasekharan[4]) of the equation

$$f(s)\Gamma\left(\frac{s}{2}\right)\pi^{-s/2} = g(1-s)\Gamma\left(\frac{1-s}{2}\right)\pi^{-(1-s)/2}. \tag{13}$$

Suppose the sequence $\{\lambda_n\}$ satisfies the condition $\lim(\lambda_{n+1} - \lambda_n) = h > 0$ and contains an "almost regular part" of some dimension r. Then $f(s)$ has the form

$$f(s) = \alpha^s \sum_{\substack{0<a\leq 1\\ 0\leq \xi<1}} A_{a,\xi}\,\zeta(s,a,\xi).$$

Here $\alpha > 0$; a runs through a finite set of values; ξ runs through not more than r values; $A_{a,\xi}$ are constants; finally, for $\operatorname{Re} s > 1$,

$$\zeta(s,a,\xi) = \sum_{n=0}^{\infty} \frac{\exp(-2\pi i \xi n)}{(n+a)^s}, \qquad 0 < a \leq 1, \quad 0 \leq \xi < 1.$$

For the proof it is necessary to apply Theorem 3 to the integrated relation (A) from [4].

Remark 1. It is known that certain linear combinations of $\zeta(s,a,\xi)$ do indeed satisfy equation (13).

Remark 2. Results analogous to Theorem 4 can be obtained for solutions of the equation

$$f(s)P(s)\Gamma\left(\frac{s}{2}\right)\pi^{-s/2} = g(1-s)Q(s)\Gamma\left(\frac{1-s}{2}\right)\pi^{-(1-s)/2},$$

where $P(s)$ and $Q(s)$ are polynomials.

I take this opportunity to express my deep gratitude to Prof. A. O. Gelfond and Prof. I. K. Andronov.

Received
7 XII 1959

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