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Abstract

Full Text

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Rotation of Weakly Continuous Vector Fields

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For mappings in bicompact topological spaces, Leray [1] introduced the concept of the full index of solutions. On its basis we introduce the concept of the rotation of a weakly continuous vector field $x - F(x)$ in a Hilbert space, where $F(x)$ maps every weakly convergent sequence into a sequence that is likewise weakly convergent, and we indicate some applications of this concept.

1. Denote by H a real separable Hilbert space and by H_n and H^m its mutually orthogonal subspaces, where H_n has dimension $n \leqslant m$, while H^m has defect m; let P_n and P^m be the operators of orthogonal projection onto these subspaces. Consider the operator

$$Jx = -P_n x + P^m x.$$

The set $H(J)$ of elements x satisfying the inequality

$$(Jx, x) \leqslant r^2, \quad r > 0, \tag{1}$$

will be called a J-domain of the space H. In particular, when $m = 0$ we obtain a ball; when $n = 0$, $m > 0$, a cylinder; when $n = m > 0$, a hyperbolic domain. In the usual way, a weak topology is introduced in the domain (1). Sets open in the weak topology of the J-domain will be denoted by U. For example, the set cut out from the J-domain by the open cylinder

$$\|P_m x\| < \rho, \tag{2}$$

is an open set in $H(J)$, which we shall call an open J-ball and denote by $T(J)$.

We shall say that a weakly continuous vector field $x - F(x)$ is defined on the closure $\overline{U}$ of a bounded open set U if the weakly continuous operator $F(x)$ is defined on $\overline{U}$ and takes values in the J-domain:

$$(JF(x), F(x)) \leqslant r^2. \tag{3}$$

Two fields $x - F_0(x)$ and $x - F_1(x)$ will be called weakly homotopic if there exists a field $x - \Phi(t, x)$, weakly continuous in (t, x), which for $t = 0$ and $t = 1$ coincides respectively with the first and the second.

Suppose that on the boundary $\dot{U}$ the field has no zero vectors. The sets U and FU are bounded and are contained in some closed J-ball $\overline{T}(J)$. We equip $\overline{T}(J)$ with the weak topology.

Lemma. *The set $\overline{T}(J)$ is a Hausdorff connected bicompact space with the first axiom of countability and is weakly continuously deformable onto itself to a point.*

The set U is an open set in $\overline{T}(J)$; on it is defined the full index (in the space $\overline{T}(J)$) of the operator $F(x)$ in the sense

Leray ([1]), which depends neither on the choice of $\bar{T}(J)$, nor on the method of extending the mapping from the boundary $\dot{U}$ to U, and therefore we call it the rotation.

Let us formulate, in the form of a theorem, the main result on rotation that follows from Leray' s theorems.

Theorem 1. *Suppose that on the boundary $\dot{U}$ of a bounded open set U a weakly continuous field does not vanish.*

Then the rotation γ of the field $x - F(x)$ on the boundary $\dot{U}$ is an integer: positive, negative, or zero. The rotation of the field is determined only by the values of $F(x)$ on the boundary $\dot{U}$ and does not depend on the extension of the field to U. Weakly homotopic fields have the same rotation, if the connecting field $x - \Phi(t, x)$, for $0 \leq t \leq 1$, $x \in \dot{U}$, does not vanish. If $\bar{U}$ consists of the sum of sets $\bar{U}_\alpha$, $U_\alpha \cap U_\beta = 0$, and on the boundaries $\dot{U}_\alpha$ the rotations γ_α of the field $x - F(x)$ are defined, then $\gamma = \sum \gamma_\alpha$.

In particular, if U_α contains the single fixed point $x_\alpha = F(x_\alpha)$, then the rotation γ_α is called its index. When U contains a finite number of fixed points x_α, the rotation γ of the field is equal to the sum of their indices.

2. One of the basic problems in the study of vector fields is the computation of their rotation. We note that a number of general theorems applicable in our realization were established by Leray. Below the rotations of several concrete fields are computed.

In finite-dimensional space, and also in a Banach space for a completely continuous field, the concept of rotation was studied in detail by M. A. Krasnosel' skii ([2]). In the case where both rotations are defined, they coincide.

Theorem 2. *Suppose that on the boundary of an open set in Euclidean space a vector field without zero vectors is given.*

Then the rotations of the field in both definitions are the same.

Let us now take as U the open J-ball $T(J)$ or the weak neighborhood of zero $U(\theta)$.

Theorem 3. *Suppose that on $\bar{T}(J)$ (or on $\bar{U}(\theta)$) a weakly continuous field $x - F(x)$ is given, without zero vectors on the boundary $\dot{T}(J)$ (respectively $\dot{U}(\theta)$), and moreover $F(\bar{T}(J)) \subset \bar{T}(J)$ (or $F(\bar{U}(\theta)) \subset \bar{U}(\theta)$).*

Then the rotation of the field on $\dot{T}(J)$ (respectively on $\dot{U}(\theta)$) is equal to $+1$.

From this follows the following generalization of the Schauder–Tikhonov principle:

Theorem 4. *A weakly continuous mapping $F(x)$ of the set $\bar{T}(J)$, or $\bar{U}(\theta)$, or the ball $\|x\| \leq r^2$ into itself has a fixed point $x = F(x)$.*

Recall that the field $x - F(x)$ is called odd if $F(-x) = -F(x)$.

Theorem 5. *Let U be a bounded open connected centrally symmetric set, star-shaped with respect to the point θ (for example $T(J)$), and suppose that on $\bar{U}$ a weakly continuous field $x - F(x)$ is given, which is odd on the boundary $\dot{U}$ and has no zero vectors on it.*

Then the rotation of the field is odd.

We note that for completely continuous fields a similar assertion was proved by M. A. Krasnosel' skii ([2]), and for finite-dimensional fields it was first observed by Borsuk that the rotation is nonzero.

For computing the rotation of a field on the boundary of the J-ball

$$\|P_{mx}\| = \rho, \qquad (Jx, x) \leq r^2, \tag{4}$$

the following is useful.

Theorem 6. *If $P_{mF}(x) \neq P_{mx}$ on the boundary (4), then the rotation of the field $x - F(x)$ on the boundary (4) is equal to the rotation of the finite-dimensional field $x - P_{mF}(x)$ on the sphere $\|x\| = \rho$ of the space H_m.*

With the aid of Theorem 6, one computes the rotation of the field $x - \lambda Ax$, where A is a linear bounded self-adjoint operator and λ is its regular value. It is known that the operator A decomposes into the sum of two orthogonal components $A = A_E + A_G$, where A_E carries the purely point spectrum of the operator A, while A_G is continuous. We shall also assume that $\|A_G\| \leq |\lambda|^{-1}$ and that there is only a finite number of eigenvalues (counting their multiplicities) of the operator A that exceed $|\lambda|^{-1}$ in absolute value.

Under our assumptions, the field $x - \lambda Ax$ has only one fixed point, $x = \theta$.

Theorem 7. *The index of the fixed point $x = \theta$ of the field $x - \lambda Ax$ is equal to $(-1)^n$, where n is the sum of the multiplicities of the eigenvalues of the operator A whose absolute values exceed $|\lambda|^{-1}$.*

In particular, if $x - \lambda Ax$ is a completely continuous field and λ is not an eigenvalue, then all the conditions of the theorem are satisfied, and in this case the rotation of the weakly continuous field coincides with the rotation of the completely continuous one.

3. The construction of the rotation of a weakly continuous field given in a j-domain admits a generalization by generalizing the notion of a J-domain.

Let two functionals $\varphi(x)$ and $\psi(x)$ be given on the space H, satisfying the following conditions:

1) $\varphi(x)$ and $\psi(x)$ are weakly upper semicontinuous functionals, i.e., for $x \overset{\text{sl}}{\to} x_0$,
$$\lim \varphi(x) \geq \varphi(x_0), \qquad \lim \psi(x) \geq \psi(x_0).$$
Consider the φ-domain $H(\varphi)$
$$\varphi(x) \leq r^2 \tag{5}$$
and in it the ψ-ball $T(\psi)$
$$\psi(x) < \rho^2. \tag{6}$$

2) There exists a weakly continuous deformation $p(t,x)$, $0 \leq t \leq 1$, $p(1,x) = \theta$, $p(0,x) = x$, of the domain (5), such that $|\varphi(p(t,x))|$ and $|\psi(p(t,x))|$ are strictly decreasing functions of t.

3) The ψ-ball (6) is a bounded set in H.

4) Every ball of the space H is contained in some domain $\psi(x) \leq \rho^2$.

Introduce the weak topology in $H(\varphi)$ and $\overline{T}(\psi)$; let

5) The φ-domain $H(\varphi)$ have a closed covering $\{V\}$, containing, together with V, also their finite intersections (nonempty), and such that every neighborhood $U(x_0)$ (in $H(\varphi)$) contains such a V that x_0 lies inside V; the intersection of any V with any $\overline{T}(\psi)$, if nonempty, is weakly continuously deformable into itself to a point.

Let U be a bounded open set in $H(\varphi)$, and suppose that on its closure $\overline{U}$ there is defined a weakly continuous vector field $x - F(x)$, with no zeros on the boundary $\dot{U}$, and, moreover,
$$\varphi(F(x)) \leq r^2, \qquad x \in \overline{U}. \tag{7}$$
Then the rotation of the field on the boundary $\dot{U}$ is defined, and Theorem 1 is valid. The other theorems are preserved under additional assumptions.

4. As one of the possible applications, we give the formulation of a theorem on the existence of an implicit function.

Let X, V, Z be Hilbert spaces, and let the operator
$$F(x,y) = F(x,\theta) + R(x)y + \Phi(x,y) \tag{8}$$
be defined for $(x,y) \in M \times \overline{T}(\psi)$, where M is a bounded set in X, and $\overline{T}(\psi)$ is the weak closure of the open ψ-ball (6) of the φ-domain (5) of the space—

of V and acts in Z. Consider such a system $U_\alpha(\theta)$ of neighborhoods (in all of V) that $\bigcap U_\alpha(\theta) = \theta$. We shall denote by λU_α the product of a neighborhood by the number λ.

Theorem 8. Let $F(x,\theta)+\Phi(x,y)$ be weakly continuous on $M\times\overline{T}(\psi)$; let $[R(x)]^{-1}z$ exist and be weakly continuous with respect to $(x,z)\in M\times Z$;

$$[R(x)]^{-1}[F(x,\theta)+\Phi(x,y)]\in\overline{T}(\psi),\quad (x,y)\in M\times\dot{T}(\psi); \tag{9}$$

$$[R(x)]^{-1}[\Phi(x,y')-\Phi(x,y'')]\in\lambda U_\alpha(\theta), \tag{10}$$

as soon as

$$y'-y''\in\lambda\dot{U}_\alpha(\theta),\quad x\in M,\quad \lambda>0.$$

Then there exists a solution $y=y(x)$, $x\in M$, of the equation $F(x,y)=0$, belonging to $\overline{T}(\psi)$ and depending weakly continuously on x.

Let us note that condition (10) may be replaced by the following:

$$\big\|[R(x)]^{-1}[\Phi(x,y')-\Phi(x,y'')]\big\|<\|y'-y''\|,\quad y'-y''\in\dot{U}_\alpha(\theta),\quad x\in M.$$

5. Weakly continuous operators satisfying conditions (3) or (7) arise naturally when $H(J)$ or $H(\varphi)$ is mapped by means of trajectories of differential equations in Hilbert space. Under the condition of time-periodicity of the right-hand side, each fixed point of the indicated transformation corresponds to a periodic solution. The theorems on rotations given above make it possible not only to prove the existence of periodic solutions, but in some cases also to estimate their number.

6. We note that some of the results remain valid for the space of functionals over a separable Banach space.

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References

1. J. Leray, *J. de Math.*, **24**, 201 (1945).

2. M. A. Krasnosel' skii, *Topological Methods in the Theory of Nonlinear Integral Equations*, Moscow, 1956.

Note: Figure translations are in progress. See original paper for figures.