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Abstract

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PHYSICS

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ON A MATHEMATICAL PROBLEM CONNECTED WITH THE THEORY OF LONGITUDINALLY POLARIZED PARTICLES

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1. The energy operator of a free electron $\vec{H} = \alpha\mathbf{p} + m\beta$ $(c = 1)$ has, for a given value of the momentum $\mathbf{p}$, doubly degenerate eigenvalues $E = \pm\sqrt{m^2 + p^2}$ (m is the electron mass), and the eigenfunctions ψ_μ $(\mu = \pm 1)$ of the electron $(E = \sqrt{m^2 + p^2})$ may be chosen so that the projection of the spin onto the direction of motion

$$\frac{\vec{\Sigma}\mathbf{p}}{p}\,\psi_\mu = \mu\psi_\mu \qquad \left(\mu = \pm 1;\ \vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}\right)$$

has a definite value. The states ψ_1 and ψ_{-1} correspond to polarizations along the direction of motion and against it. The positron has analogous properties.

In recent years the attention of physicists ([1]) has been attracted to the theory of the longitudinally polarized neutrino. The neutrino energy operator $H = \vec{\sigma}\mathbf{p}$ has, for a given $\mathbf{p}$, nondegenerate levels $E = \pm p$, and the neutrino possesses one definite polarization. The polarizations of the antineutrino and the neutrino are opposite. Let us note that the electron energy operator can be represented in the form

$$H = \vec{\sigma}\mathbf{p} \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + mI \times \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where the cross denotes the product.

In this connection, it is of interest to construct an operator satisfying the following conditions:

I. The operator H is representable in the form

$$H = \vec{\sigma}\mathbf{p} \times A + mI \times B, \tag{1}$$

where A and B are some Hermitian matrices, and I is the identity matrix of order 2.

II. For a given value of the momentum $\mathbf{p}$, the operator H has two eigenvalues $E = \pm\sqrt{m^2 + p^2}$, and these values are nondegenerate.

III. The eigenfunctions $\psi_\pm$ of the operator H are simultaneously eigenfunctions of the projection of the spin operator onto the direction of the momentum, and

$$\frac{\vec{\Sigma}\mathbf{p}}{p}\,\psi_\pm = \pm\psi_\pm, \tag{2}$$

where $\vec{\Sigma} = \vec{\sigma} \times J$, and J is the identity matrix of the space in which the operators A and B act.

Conditions II, III express the longitudinal polarization of particles of mass $m \geqslant 0$.

2. To obtain a Hamiltonian satisfying conditions I—III, we shall consider the operator $\hat{H} = pJ_0 + mT$, where

$$J_0 = \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}, \qquad T = \begin{pmatrix} 0 & \sqrt{1/2} & 0 & 0 & 0 & \dots \\ \sqrt{1/2} & 0 & 1/2 & 0 & 0 & \dots \\ 0 & 1/2 & 0 & 1/2 & 0 & \dots \\ 0 & 0 & 1/2 & 0 & 1/2 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \tag{3}$$

In paper (2) it was shown that for $p > 0$ the operator $\hat{H}$ has the only eigenvalue $E = \sqrt{m^2 + p^2}$, and for $p < 0$ the eigenvalue $E = -\sqrt{m^2 + p^2}$. The desired operator H is defined by the equality

$$H = \vec{\sigma}\mathbf{p} \times J_0 + mI \times T. \tag{4}$$

Let us find the eigenvalues of the operator (4). The relation $H\psi = E\psi$, where

$$\psi = \{u_n\}_{n=0}^{\infty} \left(\sum_{n=0}^{\infty} u_n^* u_n < \infty \right)$$

is an element whose components are the spinors u_n $(n = 0, 1, 2, ...)$, is equivalent to the system of equations

$$\vec{\sigma}\mathbf{p}u_0 + \frac{m}{\sqrt{2}}u_1 = E \cdot u_0,$$

$$\frac{m}{\sqrt{2}}u_0 + \frac{m}{2}u_2 = E \cdot u_1, \tag{5}$$

$$\frac{m}{2}(u_{n-1} + u_{n+1}) = E \cdot u_n \qquad (n = 2, 3, ...).$$

Multiplying both sides of the equations on the left by the spinor u_+^*, satisfying the condition $u_+^*\vec{\sigma}\mathbf{p} = pu_+^*$, we obtain

$$p\xi_0 + \frac{m}{\sqrt{2}}\xi_1 = E \cdot \xi_0,$$

$$\frac{m}{\sqrt{2}}\xi_0 + \frac{m}{2}\xi_2 = E \cdot \xi_1, \tag{6}$$

$$\frac{m}{2}(\xi_{n-1} + \xi_{n+1}) = E \cdot \xi_n \qquad (n \geqslant 2).$$

If $\xi_0 \neq 0$, then, in accordance with the results of paper ([2]), the eigenvalue is $E = \sqrt{m^2 + p^2}$. If, however, $\xi_0 = 0$, then $\eta_0 = u_-^* u_0 \neq 0$, where u_-^* is a spinor satisfying the condition $u_-^*\vec{\sigma}\mathbf{p} = -pu_-^*$. In fact, from $\xi_0 = \eta_0 = 0$ it follows that $u_0 = 0$, which, according to (5), leads to the trivial solution $u_n = 0$ $(n \geqslant 0)$. When multiplying equations (5) on the left by u_-^*, the number p is replaced by $-p$ and, consequently, the eigenvalue is $E = -\sqrt{m^2 + p^2}$.

It is easy to see that the eigenvectors of the operator $\hat{H}$ have the form

$$\psi_\pm = \{u_\pm \xi_n^\pm\}_{n=0}^\infty, \quad \text{where } \xi_0^+ = \xi_0^- = 1, \quad \xi_n^+ = \sqrt{2}\left(\frac{E-p}{m}\right)^n, \quad \xi_n^- = (-1)^n \left(\frac{E-p}{m}\right)^n$$

$$(E = \sqrt{m^2 + p^2}).$$

Requirement III is satisfied, since

$$\frac{\vec{\Sigma}\mathbf{p}}{p}\psi_\pm = \left\{\frac{\vec{\sigma}\mathbf{p}}{p}u_\pm \xi_n^\pm\right\}_{n=0}^\infty = \pm\psi_\pm.$$

It is interesting to note that, under sufficiently broad assumptions, the operator $\vec{\sigma}\mathbf{p} \times J_0 + mI \times T$ is unique—from the point of view—

up to unitary equivalence—an operator satisfying conditions I-III. Indeed, if some operator H satisfies conditions I-III and $\psi_+ = \{u_n\}_{n=0}^{\infty}$ is an eigenvector corresponding to the eigenvalue $E = \sqrt{m^2+p^2}$, then, by the condition $\vec{\sigma} p u_n = p u_n$, the spinors u_n have the form $u_n = u_+ \xi_n$ $(n \geqslant 0)$, where ξ_n are certain numbers. The equation $H\psi_+ = E\psi_+$ is rewritten in the form

$$p\vec{A}\xi + m\vec{B}\xi = E \cdot \xi \quad (\xi = \{\xi_n\}_{n=0}^{\infty}),$$

and, consequently, $E = \sqrt{m^2+p^2}$ is an eigenvalue of the operator $mB + pA$ $(p > 0)$. This operator can have no other eigenvalues, since from the equality $p\vec{A}\xi' + m\vec{B}\xi' = E' \cdot \xi'$ $(E' \neq E)$ it would follow that $\psi' = \{u_+ \xi_n'\}$ is an eigenvector of the operator H corresponding to the value $E' \neq \sqrt{m^2+p^2}$. The possibility $E' = -\sqrt{m^2+p^2}$ is excluded, since ψ' does not satisfy the condition $\frac{\vec{\Sigma} p}{p}\psi' = -\psi'$. In paper ([2]) it was shown that if A is completely continuous and B is an arbitrary bounded Hermitian operator, and $pA + mB$ has the above-indicated unique eigenvalue for all $p > 0$, then there exists a unitary transformation that simultaneously brings A and B to the form $A = J_0,\ B = T$.

3. The four eigenfunctions of the Dirac operator $\vec{\alpha} p + m\beta$ for a given value of p form a complete system of states in the corresponding space. In the case of the operator $\vec{\sigma} p \times J_0 + mI \times T$, the two stationary states $\psi_\pm$ obviously do not form a complete system in the infinite-dimensional space of elements of the form $\{u_n\}_{n=0}^{\infty}$. It can be shown ([2]) that, besides the eigenvalues $E = \pm\sqrt{m^2+p^2}$, for which the condition $\sum_{n=0}^{\infty} u_n^* u_n < \infty$ is fulfilled, the operator (4) has a fixed continuous spectrum coinciding with the segment $-m \leqslant E \leqslant m$. An interesting physical interpretation of the properties of the operator (4) was proposed by M. M. Alperin ([3]).

Let us also find an expression for the current $\mathbf{j}$ in the case of the Schrödinger equation

$$i\frac{\partial \psi}{\partial t} = -i\vec{\sigma}\nabla \times J_0 \psi + mI \times T\psi$$

with energy operator (4). After simple transformations we shall have

$$\frac{d}{dt}\int_V \psi^* \psi \, dv = -\int_V \operatorname{div}(\psi^* \vec{\sigma} \times J_0 \psi)\, dv,$$

whence follows the continuity equation $\partial\rho/\partial t + \operatorname{div}\mathbf{j} = 0$, where $\rho = \sum_{n=0}^{\infty} u_n^* u_n$ is the particle density, and $\mathbf{j} = u_0^* \vec{\sigma} u_0$ is the current. Thus, the current $\mathbf{j}$ depends only on the zeroth component u_0, and in any state for which $u_0 = 0$, the current $\mathbf{j} = 0$. It is not difficult to verify that in the stationary state

$$\psi_+ = \left(u_+,\ \sqrt{2}\left(\frac{E-p}{m}\right)u_+,\ \sqrt{2}\left(\frac{E-p}{m}\right)^2 u_+, \dots\right)$$

the current $\mathbf{j} = \frac{\mathbf{p}}{p} u_+^* u_+$, while the density $\rho = \frac{E}{p} u_+^* u_+$, and therefore the relation $\mathbf{j} = \frac{\mathbf{p}}{E}\rho = \mathbf{v}\rho$, characteristic of particles moving with velocity $\mathbf{v}$, is fulfilled. Similarly, in the state ψ_- we have $\mathbf{j} = -\mathbf{v}\rho$.

Using the operator (4), one can construct an operator $\widetilde{H}$ of a somewhat more general form

$$\widetilde{H} = \begin{pmatrix} {}^{(2)}H & \mu \\ \mu & 0 \end{pmatrix} \quad (\mu = \sqrt{m_1^2 - m_2^2}, \quad m_1 > m_2 > 0), \tag{7}$$

where H is the operator (4), in which $m = m_2$. The operator $\widetilde{H}$ acts in the space of pairs of elements $\binom{\psi_1}{\psi_2}$ and, for a given momentum $\mathbf{p}$, has 4 eigenvalues, which are determined by the equalities

$$E = \pm\sqrt{m_1^2 + p^2} \pm \sqrt{m_2^2 + p^2}. \tag{8}$$

Indeed, if

$$\begin{pmatrix} 2H & \mu \\ \mu & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = E \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

then

$$2H\psi_1 + \mu\psi_2 = E \cdot \psi_1, \qquad \mu\psi_1 = E \cdot \psi_2,$$

whence $H\psi_1 = \dfrac{1}{2}\left(E - \dfrac{\mu^2}{E}\right)\psi_1$. Since $\psi_1 \neq 0$, by property II of the operator H, we have

$$\frac{1}{2}\left(E - \frac{\mu^2}{E}\right) = \pm\sqrt{m_2^2 + p^2},$$

and, consequently, E is determined by the equalities (8). The corresponding eigenvectors have the form

$$\begin{pmatrix} \psi_1 \\ \dfrac{\mu}{E}\psi_1 \end{pmatrix}, \tag{9}$$

Each level (8) is nondegenerate, and the eigenvectors (9) are simultaneously also eigenvectors of the projection

$$\begin{pmatrix} \dfrac{\vec{\Sigma}\mathbf{p}}{p} & 0 \\ 0 & \dfrac{\vec{\Sigma}\mathbf{p}}{p} \end{pmatrix}$$

of the spin operator onto the direction of the momentum. We may interpret the eigenfunctions (9) as stationary states of two particles—antiparticles of masses m_1 and m_2, whose motion is referred to the coordinate system associated with the center of mass of these particles. From this point of view one may also approach the operator H considered earlier, which is obtained from $\widetilde{H}$ for $m_1 = m_2 = \frac{1}{2}m$ and with $\mathbf{p}$ replaced by $\frac{1}{2}\mathbf{p}$.

The Schrödinger equation

$$i\frac{\partial\psi}{\partial t} = (\vec{\sigma}\mathbf{p} \times J_0 + mL \times T)\psi$$

is invariant with respect to rotations of the coordinate system about the center of mass. Under reflection the momentum $\mathbf{p}$ is replaced by $-\mathbf{p}$, and, as is easy to see, particles are replaced by antiparticles.

Let us also note that the operator $\widetilde{H}$ has a fixed continuous spectrum filling the intervals

$$m_1 + m_2 \geq E \geq m_1 - m_2, \qquad -(m_1 + m_2) \leq E \leq m_2 - m_1,$$

corresponding to the bound states of a system of two particles with masses m_1, m_2.

The equation $i\dfrac{\partial\psi}{\partial t} = \widetilde{H}\psi$ describes collisions of particles with masses m_1, m_2 $(m_1 \geq m_2)$ relative to the coordinate system associated with the center of mass of these particles.

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