MATHEMATICS

G. V. VIRABYAN

1960

SovietRxiv

Abstract

Full Text

MATHEMATICS

G. V. VIRABYAN

ON THE SPECTRAL EQUIVALENCE OF TWO OPERATORS GENERATED BY ONE CLASS OF SYSTEMS OF DIFFERENTIAL EQUATIONS OF S. L. SOBOLEV TYPE

(Presented by Academician S. L. Sobolev on 24 II 1960)

Let Ω be a bounded domain of n-dimensional Euclidean space R_n, bounded by a sufficiently smooth hypersurface Γ. Consider in the domain Ω the Hilbert space H_1, which is obtained by completing the linear manifold $D_{\mathfrak{A}}$ of smooth solenoidal n-dimensional vectors whose components are square-integrable in Ω, with respect to the scalar product

$$(\mathbf{u},\mathbf{v})_1=\int_\Omega\cdots\int\{u_1\overline{v}_1+u_2\overline{v}_2+\cdots+u_n\overline{v}_n\}\,d\Omega. \tag{1}$$

In the Hilbert space H_1 consider the operator $\mathfrak{A}$, given by the formula

$$\mathbf{v}\in D_{\mathfrak{A}},\qquad \mathfrak{A}\mathbf{v}=A\mathbf{v}+B\operatorname{grad}S\mathbf{v},\qquad S\mathbf{v}=P, \tag{2}$$

where P is determined from the boundary-value problem

$$L(P)=\operatorname{div}A\mathbf{v},\qquad P\big|_\Gamma=0, \tag{3}$$

$$L=-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(b_{ij}\frac{\partial}{\partial x_j}\right)$$

is a second-order differential operator with variable coefficients of elliptic type; A and B are n-dimensional square variable matrices $A=\|a_{ij}\|$, $B=\|b_{ij}\|$ $(i,j=1,2,\ldots,n)$, with: 1) $A^2=E$; 2) B a positive-definite matrix; 3) $AB=BA$.

The operator $\mathfrak{A}$ is generated by the following class of systems of differential equations with variable coefficients of S. L. Sobolev type (1):

$$\frac{\partial^2 v_i}{\partial t^2} = \sum_{j=1}^{n} a_{ij} v_j + \sum_{j=1}^{n} b_{ij} \frac{\partial P}{\partial x_j}, \qquad \sum_{i=1}^{n} \frac{\partial v_i}{\partial x_i} = 0 \quad (i = 1, 2, \dots, n). \tag{4}$$

On the other hand, consider the Hilbert space H_2, which is obtained from the linear manifold D_Q of infinitely differentiable finite functions in the domain Ω by completing with respect to the scalar product

$$(u, v)_2 = \int_\Omega \cdots \int \sum_{i,j=1}^{n} b_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \overline{v}}{\partial x_j} \, d\Omega = \int_\Omega \cdots \int Lu \cdot \overline{v} \, d\Omega, \tag{5}$$

or, what is the same,

$$(u, v)_2^* = \int_\Omega \cdots \int \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial \overline{v}}{\partial x_j} \, d\Omega. \tag{6}$$

In the Hilbert space H_2 consider the operator Q, defined by the formula $Q = -L^{-1}M$, where L^{-1} is the operator inverse to the elliptic operator

$$L = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(b_{ij} \frac{\partial}{\partial x_j} \right), \qquad M = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(c_{ij} \frac{\partial}{\partial x_j} \right),$$

$$\|c_{ij}\| = \|a_{ij}\| \|b_{ij}\|, \quad \text{i.e. } C = AB.$$

Denote by $H_1^{(A)}$ the proper subspace of the operator $\mathfrak{A}$ corresponding to the matrix A.

By $H_1^{(1)}$ and $H_2^{(1)}$ we denote the subspaces that correspond to the discrete parts of the spectra, respectively, for the operator $\mathfrak{A}$ in $H_1 \ominus H_1^{(A)}$ and for the operator Q in H_2. Put

$$H_{\mathfrak{A}} = H_1 \ominus \{H_1^{(A)} \oplus H_1^{(1)}\}, \qquad H_Q = H_2 \ominus H_2^{(1)}. \tag{7}$$

In the work (^{1}) of R. A. Aleksandryan, a spectral equivalence of the operators $\mathfrak{A}$ and Q was established in a certain sense. In the present note we establish, in a known sense, a deeper spectral connection between these operators, namely, the following is proved:

Theorem. *In the case of a continuous spectrum, from a complete system of proper differentials of the operator* $\mathfrak{A}$ *in* $H_{\mathfrak{A}}$ *one can construct a complete system of proper differentials for the operator* Q *in* H_Q*, and conversely.*

Proof. Let $\mathbf{v}(\lambda) = (v_1(\lambda), v_2(\lambda), \dots, v_n(\lambda)) \in H_{\mathfrak{A}}$ be a proper differential for the operator $\mathfrak{A}$ in $H_{\mathfrak{A}}$; this means

$$\mathfrak{A}[\Delta\mathbf{v}(\lambda)] = \int_{\Delta} \lambda\, d\mathbf{v}(\lambda) \tag{8}$$

for all intervals Δ, i.e.

$$A[\Delta\mathbf{v}(\lambda)] + B\,\mathrm{grad}[\Delta P(\lambda)] = \int_{\Delta} \lambda\, d\mathbf{v}(\lambda), \tag{9}$$

where $L(P(\lambda)) = \mathrm{div}\, A\mathbf{v}(\lambda)$, $P(\lambda)|_{\Gamma} = 0$. Then one can prove that $P(\lambda)$ is a proper differential for the operator Q in H_Q. Conversely, if $u(\lambda)$ is a proper differential for the operator Q in H_Q, then by a direct verification one can ascertain that the vector

$$\mathbf{v}(\lambda) = AB\,\mathrm{grad}\, u(\lambda) + \int_{-\infty}^{\lambda} \mu\, dB\,\mathrm{grad}\, u(\mu) \tag{10}$$

is a proper differential for the operator $\mathfrak{A}$ in $H_{\mathfrak{A}}$. It remains to prove that by the indicated constructions we obtain all proper differentials of the operators $\mathfrak{A}$ in $H_{\mathfrak{A}}$ and Q in H_Q, i.e. to prove the so-called completeness property.

Suppose that $u(\lambda) \in H_Q$ is a proper differential for the operator Q in H_Q; then, as was shown above, the vector-function

$$\mathbf{v}(\lambda) = AB\,\mathrm{grad}\, u(\lambda) + \int_{-\infty}^{\lambda} \mu\, dB\,\mathrm{grad}\, u(\mu) \tag{11}$$

will be a proper differential for the operator $\mathfrak{A}$ in $H_{\mathfrak{A}}$. Further, let $u^*(\lambda)$ be the proper differential of the operator Q in H_Q corresponding to the proper differential $\mathbf{v}(\lambda)$ of the operator $\mathfrak{A}$, i.e.

$$Lu^*(\lambda) = \mathrm{div}\, A\mathbf{v}(\lambda); \tag{12}$$

$$u^*(\lambda)|_{\Gamma} = 0. \tag{13}$$

We shall prove that the differential subspaces $\overline{\{u(\lambda)\}}$ and $\overline{\{u^*(\lambda)\}}$, generated respectively by the eigenelement differentials $u(\lambda)$ and $u^*(\lambda)$ of the operator Q in H_Q, coincide, i.e.

$$\overline{\{u(\lambda)\}} \equiv \overline{\{u^*(\lambda)\}}. \tag{14}$$

For this, note that from (11) and (12) we have

$$Lu^*(\lambda) = \operatorname{div} B \operatorname{grad} u(\lambda) + \operatorname{div} A \left\{ \int_{-\infty}^{\lambda} \lambda \, dB \operatorname{grad} u(\lambda) \right\} =$$

$$= -Lu(\lambda) - \int_{-\infty}^{\lambda} \mu \, dM u(\mu) = -Lu(\lambda) - M \left\{ \int_{-\infty}^{\lambda} \lambda \, du(\lambda) \right\}. \tag{15}$$

Hence

$$u^*(\lambda) = -u(\lambda) + Q \left\{ \int_{-\infty}^{\lambda} \lambda \, du(\lambda) \right\} = (Q^2 - E)u(\lambda). \tag{16}$$

Thus we must prove that

$$\overline{\{(Q^2 - E)u(\lambda)\}} \equiv \overline{\{u(\lambda)\}}. \tag{14*}$$

It is obvious that

$$\overline{\{(Q^2 - E)u(\lambda)\}} \subset \overline{\{u(\lambda)\}}. \tag{17}$$

Next, from

$$(v, (Q^2 - E)u(\lambda))_2 = 0 \tag{18}$$

it follows that

$$((Q^2 - E)v, u(\lambda)) = 0, \tag{19}$$

i.e.

$$(Q^2 - E)v = 0, \tag{20}$$

and since the numbers ±1 are not eigenvalues of the operator Q in H_Q, we have

$$v \equiv 0, \tag{21}$$

and therefore (14) is proved.

Thus, we have proved that, from a complete system of eigenelement differentials of the operator $\mathfrak{A}$ in $H_{\mathfrak{A}}$, one can construct a complete system of eigenelement differentials for the operator Q in H_Q.

To prove the converse assertion, suppose that $v(\lambda)$ is some eigenelement differential for the operator $\mathfrak{A}$ in $H_{\mathfrak{A}}$. Let $P(\lambda)$ be the eigenelement differential for the operator Q in H_Q corresponding to $v(\lambda)$. Further, let

$$v^*(\lambda) = AB \operatorname{grad} P(\lambda) + B \operatorname{grad} \int_{-\infty}^{\lambda} \mu \, dP(\mu)$$

be the eigenelement differential for the operator $\mathfrak{A}$ in $H_{\mathfrak{A}}$ corresponding to the eigenelement differential $P(\lambda)$ for the operator Q in H_Q. We show that the differential subspaces $\overline{\{v(\lambda)\}}$ and $\overline{\{v^*(\lambda)\}}$, generated respectively by the eigenelement differentials $v(\lambda)$ and $v^*(\lambda)$ for the operator $\mathfrak{A}$ in $H_{\mathfrak{A}}$, coincide, i.e.

$$\overline{\{v(\lambda)\}} \equiv \overline{\{v^*(\lambda)\}}. \tag{22}$$

For this, let us note that

$$B \operatorname{grad} P(\lambda) = \mathfrak{A}\mathbf{v}(\lambda) - A\mathbf{v}(\lambda). \tag{23}$$

Further,

$$\mathbf{v}^*(\lambda) = A\mathfrak{A}\mathbf{v}(\lambda) - A^2\mathbf{v}(\lambda) + \int_{-\infty}^{\lambda} \mu \, d\mathfrak{A}\mathbf{v}(\mu) - \int_{-\infty}^{\lambda} \mu \, dA\mathbf{v}(\mu) =$$

$$= A\mathfrak{A}\mathbf{v}(\lambda) - \mathbf{v}(\lambda) + \mathfrak{A}^2\mathbf{v}(\lambda) - A\mathfrak{A}\mathbf{v}(\lambda) = (\mathfrak{A}^2 - E)\mathbf{v}(\lambda). \tag{23*}$$

Thus, we must prove that $\overline{\{\mathbf{v}(\lambda)\}} \equiv \overline{\{(\mathfrak{A}^2 - E)\mathbf{v}(\lambda)\}}$. Obviously,

$$\overline{\{(\mathfrak{A}^2 - E)\mathbf{v}(\lambda)\}} \subset \overline{\{\mathbf{v}(\lambda)\}}. \tag{24}$$

Further, from the fact that

$$(\mathbf{w}, (\mathfrak{A}^2 - E)\mathbf{v}(\lambda))_1 = 0 \tag{25}$$

for all λ, it follows that

$$((\mathfrak{A}^2 - E)\mathbf{w}, \mathbf{v}(\lambda))_1 = 0, \tag{26}$$

whence

$$(\mathfrak{A}^2 - E)\mathbf{w} = 0, \tag{27}$$

and since the numbers ±1 are not eigenvalues of the operator $\mathfrak{A}$ in $H_{\mathfrak{A}}$, we obtain

$$\mathbf{w} \equiv 0, \tag{28}$$

i.e., the subspaces $\overline{\{\mathbf{v}(\lambda)\}}$ and $\overline{\{(\mathfrak{A}^2 - E)\mathbf{v}(\lambda)\}}$ coincide. This proves the theorem completely.

Computing Center
Academy of Sciences of the Armenian SSR

Received
25 January 1960

REFERENCES

[1] R. A. Aleksandryan, *DAN*, **131**, No. 3 (1960).
[2] G. V. Virabyan, *DAN*, **128**, No. 1 (1959).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.