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Abstract

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ON STABLE AND UNSTABLE TRAJECTORIES OF PROPORTIONAL NAVIGATION

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The cybernetic method of investigating proportional navigation [1] has until now been limited to the case when the navigation constant $b = 2$. It was considered established that a closed-form solution even of the simplest kinematic problem for $b > 2$ is impossible to obtain. Locke [2] writes that a solution of the equations of motion in closed form can be found only for $b = 2$; if $b \neq 2$, numerical integration must be used. Therefore, in engineering practice the method of modeling on electronic computing devices was used. In this way, however, it is impossible to evaluate the entire problem as a whole and to obtain general conclusions independent of the specific values of the parameters and the initial conditions of motion. In addition, numerical integration of the differential equations of the perturbed motion did not make it possible to say anything about the stability of the motion. Proportional navigation is a "complex" interdependent system [3], which does not admit investigation by the method of "varying the factors one at a time." Precisely for this reason the more difficult path of analytical investigation of the problem cannot be rejected. Let us consider proportional navigation for arbitrary integer values of the navigation constant.

The differential equations of the ideal motion of an axisymmetric object in the horizontal plane [1] will be

$$mv\dot{\psi} = (T + c_L v^2)\alpha; \qquad I_z\ddot{\varphi} = -k_1(v)\beta - k_2(v)\dot{\varphi} + k_3(v)\alpha;$$

$$\varphi = \eta - 90^\circ + \alpha - \gamma; \qquad \psi + \gamma = \eta; \qquad \dot{\psi} = b\dot{\eta};$$

$$\dot{a} = v_s \cos\eta - v\cos\gamma; \qquad a\dot{\eta} = v\sin\gamma - v_s \sin\eta. \tag{1}$$

(the notation is the same as in (1)).

We shall show that the system of the last four equations (1) permits a closed-form solution for arbitrary integer values of the navigation constant. Let us write this system in the form

$$\dot{a} = v_s[\cos\eta - p\cos(b-1)(\eta-\varepsilon_0)] \equiv v_s F(\eta); \tag{2}$$

$$a\dot{\eta} = -v_s[\sin\eta + p\sin(b-1)(\eta-\varepsilon_0)] \equiv -v_s f(\eta), \tag{3}$$

where $p = v/v_s$; $\varepsilon_0 = (b\eta_0 - \psi_0)/(b-1)$, and the subscript zero corresponds to the initial values of the variables. Integrating this system for the value $b = 3$, we find the trajectory equation

$$\frac{a}{a_0} = \left[\frac{p\sin 2(\eta-\varepsilon_0)+\sin\eta}{p\sin 2(\eta_0-\varepsilon_0)+\sin\eta_0}\right]^{1/2} \prod_{i=1}^{4} \left|\frac{\operatorname{tg}\frac{1}{2}(\eta-\varepsilon_0) - \operatorname{tg}\frac{1}{2}(\eta_i-\varepsilon_0)}{\operatorname{tg}\frac{1}{2}(\eta_0-\varepsilon_0) - \operatorname{tg}\frac{1}{2}(\eta_i-\varepsilon_0)}\right|^{-3B_i \operatorname{ctg}\varepsilon_0}, \tag{4}$$

where

$$B_i = \frac{\cos\eta_i \sec^2\frac{1}{2}(\eta_i-\varepsilon_0)}{2[2p\cos 2(\eta_i-\varepsilon_0)+\cos\eta_i]};$$

η_i are the roots of the equation

$$f(\eta) = p\sin 2(\eta-\varepsilon_0) + \sin\eta = 0.$$

For $p > 1$ the latter equation has, in the interval $0 \le \eta < 2\pi$, four real roots.

For particular values of ε_0, the trajectory equation takes a simpler form. Thus, for $b = 3$, $\varepsilon_0 = 0$, we have

$$\frac{a}{a_0} = \left|\frac{1-\cos\eta}{1-\cos\eta_0}\right|^{\frac{p-1}{2(1+2p)}} \left|\frac{1+\cos\eta}{1+\cos\eta_0}\right|^{\frac{p+1}{2(2p-1)}} \left|\frac{1+2p\cos\eta}{1+2p\cos\eta_0}\right|^{\frac{2(p^2-1)}{4p^2-1}}.$$

Integrating the system (2), (3) for the case $b = 4$, we obtain the trajectory equation

$$\frac{a}{a_0}\left[\frac{p\sin 3(\eta-\varepsilon_0)+\sin\eta}{p\sin 3(\eta_0-\varepsilon_0)+\sin\eta_0}\right]^{1/3} \prod_{i=1}^{3} \left|\frac{\operatorname{tg}(\eta-\varepsilon_0) - \operatorname{tg}(\eta_i-\varepsilon_0)}{\operatorname{tg}(\eta_0-\varepsilon_0) - \operatorname{tg}(\eta_i-\varepsilon_0)}\right|^{-\frac{4}{3}A_i} = \frac{\sin\varepsilon_0}{p-\cos\varepsilon_0}, \tag{5}$$

where

$$A_i = \frac{(p-\cos\varepsilon_0)}{\sin\varepsilon_0} \frac{\cos\eta_i \sec^2(\eta_i-\varepsilon_0)}{[3p\cos 3(\eta_i-\varepsilon_0)+\cos\eta_i]};$$

η_i are the roots of the equation $f(\eta) = 0$.

The exact solution for arbitrary integer values of the navigation constant splits into two cases: b odd and b even. The system (2), (3) is reduced to the form

$$\ln \frac{a}{a_0} = \frac{1}{b-1} \ln \frac{p \sin(b-1)(\eta - \varepsilon_0) + \sin \eta}{p \sin(b-1)(\eta_0 - \varepsilon_0) + \sin \eta_0} - \frac{b}{b-1} \int_{\eta_0}^{\eta} \frac{\cos \eta \, d\eta}{p \sin(b-1)(\eta - \varepsilon_0) + \sin \eta}. \tag{6}$$

In the case of b odd, $b = 2m+1, \ m = 0, 1, 2, ...$, the integral in (6) takes the form

$$I = \int_{z_0}^{z} \frac{2[\cos \varepsilon_0 (1 - z^2) - \sin \varepsilon_0 \cdot 2z](1+z^2)^{2m-2}}{pQ_{4m-1}(z) + (1-z^2)(1+z^2)^{2m-1} \sin \varepsilon_0 + 2z(1+z^2)^{2m-1} \cos \varepsilon_0} \, dz, \tag{7}$$

where $z = \mathrm{tg}^{1/2}(\eta - \varepsilon_0)$; Q_{4m-1} is a polynomial of degree $4m-1$. Thus, the denominator contains a polynomial of degree $4m$. If its roots are $z_1, z_2, \dots, z_{4m}$, then the subsequent reasoning proceeds in the same way as in the case $b = 3$.

For b even, $b = 2m+2$, where $m = 0, 1, 2, ...$, the integral (6) is equal to

$$I = \int_{z_0}^{z} \frac{(\cos \varepsilon_0 - z \sin \varepsilon_0)(1+z^2)^{m-1} \, dz}{pQ_{2m+1}(z) + (\sin \varepsilon_0 + z \cos \varepsilon_0)(1+z^2)^m}, \tag{8}$$

where $z = \mathrm{tg}(\eta - \varepsilon_0)$. The denominator contains a polynomial of degree $(2m+1)$. If its roots are $z_1, z_2, \dots, z_{2m+1}$, then we proceed further in the same way as in the case $b = 4$.

After the trajectory equation $a = f(\eta)$ has been obtained, using the system (1), we easily find $\dot{\eta}, \dot{\psi}, \alpha, \gamma, \varphi, \beta$ as functions of the rotation angle of the line of sight η.

The curvature of the trajectory, defined in polar coordinates a, η, will be

$$K = \frac{\dot{\eta}\,[\dot{a}^2 + (a\dot{\eta})^2] + (a\dot{\eta})\dot{\eta}\dot{a} - a\dot{\eta}\ddot{a}}{[\dot{a}^2 + (a\dot{\eta})^2]^{3/2}\,\mathrm{sgn}(ds/dt)}, \tag{9}$$

or, taking into account (2), (3),

$$K = -\frac{f(\eta)\, p^b \{p - \cos[(b-1)(\eta - \varepsilon_0) + \eta]\}}{a\,[f^2(\eta) + F^2(\eta)]^{3/2}\,\mathrm{sgn}(\dot{s})}. \tag{10}$$

Since $f(\eta)$ always has a constant sign (between roots), $\mathrm{sgn}(\dot{s}) = \mathrm{sgn}(\dot{\eta})$, for the arc increases with increasing η; hence, for $p > 1$ the curvature $K < 0$, and the

Figure 1. Distribution of stable and unstable roots in the case $b = 4$

Figure 1: Figure 1. Distribution of stable and unstable roots in the case $b = 4$

curve is always convex with respect to the straight line joining the object to the target.

Let us consider the behavior of the object near the target, when $a \to 0$ and the speed of the object is greater than the speed of the target ($p > 1$). From (3) it follows that for η_i,

being a root of the equation

$$f(\eta) = \sin\eta + p\sin(b-1)(\eta - \varepsilon_0) = 0, \tag{11}$$

there will occur either exact coincidence of the object and the target (interception) at $a = 0$, or $\dot{\eta} = 0$.

Equation (11) has, in the interval $0 \le \eta < 2\pi$, exactly $2(b-1)$ real roots, located in the intervals between the maxima and minima of the function $\sin(b-1)(\eta-\varepsilon_0)$, since the sign of $f(\eta)$ at these points is determined by the sign of the second term. The roots (denoted in increasing order by $\eta_1, \dots, \eta_{2(b-1)}$), constructed at any point of the plane, divide the region about the point into $2(b-1)$ angles, whose boundaries are the straight lines $\eta = \eta_i$.

Fig. 1. Distribution of stable and unstable roots in the case $b = 4$

Let us call a root η_i stable if, when η is in the sector adjacent to it, as time increases $\eta \to \eta_i$. The function $f(\eta)$ does not change sign between roots. For stable roots it must be that $\dot{\eta} > 0$ for $\eta < \eta_i$ and $\dot{\eta} < 0$ for $\eta > \eta_i$. But the signs of $\dot{\eta}$ and $f(\eta)$ are opposite. Consequently, for stable roots $f(\eta) < 0$ when $\eta < \eta_i$ and $f(\eta) > 0$ when $\eta > \eta_i$. Thus, for stable roots $f'(\eta_i) > 0$, and for unstable roots $f'(\eta_i) < 0$. Note that $f'(\eta_i)$ cannot vanish and, therefore, (11) has no multiple roots. Hence $f'(\eta_i)$ has alternating signs for $\eta_1, \eta_2, \dots, \eta_{2(b-1)}$. Consequently, stable and unstable roots alternate. Stable roots correspond to approach to the target, and unstable roots to moving away from the target. This follows from the fact that the roots η_i of equation (11), when substituted into $f'(\eta_i)$ and into the expression $F(\eta)$, give them different signs in all cases.

The general scheme of the distribution of the roots η_i and the signs of $\dot{a}$ (for a given ε_0) has the form shown in Fig. 1, where the construction is made for $b = 4$, and the first root η_1 is, for definiteness, taken to be stable. In approaching along a rectilinear trajectory $\eta = \eta_i$, where $f'(\eta_i) > 0$, stability holds "in the large," since $f'(\eta)$ and, consequently, $\dot{\eta}$ do not change sign between the roots (11). Thus any perturbation that does not carry the motion beyond the neighboring root will decay, tending to zero, since at interception η is always equal to η_i. Recession along a straight line, where $f'(\eta) < 0$, is unstable "in the small."

Fig. 2. Stability boundary in the case $b = 4,\ n = 4$

Figure 2: Fig. 2. Stability boundary in the case $b = 4,\ n = 4$

Let us consider the motion of the object near the stable roots. Expanding the right-hand sides of (2) and (3) in series near the stable roots, we have

$$\dot{a} \simeq v_s[\cos\eta_i - p\cos(b-1)(\eta_i - \varepsilon_0)] = v_{sF}(\eta_i); \tag{12}$$

$$a\dot{\eta} \simeq -v_s[\cos\eta_i + p(b-1)\cos(b-1)(\eta_i - \varepsilon_0)](\eta - \eta_i) = -v'_{sf}(\eta_i)(\eta - \eta_i). \tag{13}$$

In the expansion in series only the principal terms have been retained: in (12) only the zero-order term, and in (13) only the first, since the zero-order term is equal to zero. Integrating this system of equations, we find

$$\dot{\eta} = -\frac{v_s}{a_0} f'(\eta_i) \left| \frac{\eta - \eta_i}{\eta_0 - \eta_i} \right|^{[F(\eta_i) + f'(\eta_i)]/f'(\eta_i)}. \tag{14}$$

Thus, the behavior of $\ddot{\eta}$ as $\eta \to \eta_i$, and consequently also the behavior of $\dot{\psi}$, of the normal acceleration, and of the overload, is determined by the sign of

$$F(\eta_i) + f'(\eta_i) = 2\cos\eta_i + p(b-2)\cos(b-1)(\eta_i - \varepsilon_0), \tag{15}$$

since the denominator of the exponent in (14) is always positive. Computing further the following derivatives with respect to η, we find the general form of the numerator in the exponent

$$\theta(\eta) = f'(\eta_i) + (n-1)F(\eta_i) = n\cos\eta_i + p(b-n)\cos(b-1)(\eta_i - \varepsilon_0). \tag{16}$$

Fig. 2. Stability boundary in the case
$b = 4,\ n = 4$

The sign of this function determines, at interception: for $n = 1$, the stability of the roots; for $n = 2$, the behavior of the normal acceleration, the angular velocity of the line of sight, and the overload; for $n = 3$, the behavior of the angular velocity of the object; for $n = 4$, the behavior of the rudder deflection angle; for $n = 5$, the behavior of the rudder angular velocity; for $n = 6$, the change in the rudder angular acceleration.

Fig. 3. Stability boundary in the case
$b = 4,\ n = 3$

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Figure 3: Fig. 3. Stability boundary in the case $b = 4,\ n = 3$

Fig. 4. Stability boundary in the case $b = 4,\ n = 5$. Each branch of the curve determines the stability boundary with respect to the rudder angular velocity for the given η_i

Figure 4: Fig. 4. Stability boundary in the case $b = 4,\ n = 5$. Each branch of the curve determines the stability boundary with respect to the rudder angular velocity for the given η_i

A positive sign of $\theta(\eta_i)$ for a given n means that the corresponding variable tends to zero at interception (the overload tends to unity). A negative sign of $\theta(\eta_i)$ predetermines, near the target, an unbounded growth of the corresponding variable. If $\theta(\eta_i)$ is positive for a given n, then all functions $\theta(\eta_i)$ for smaller n are certainly positive. For $b \geq 2n$, the function $\theta(\eta_i)$ near the target is positive for any values of p, ε_0, η_i. Thus, for $b \geq 12$, interception is possible for any velocity ratio p, for any initial conditions, and from any direction. If $b < 2n$, then, equating (16) to zero and using (11), we eliminate η and find the stability boundary for the corresponding variables (depending on the chosen n). Such curves are presented in Figs. 2, 3, and 4. If $b < n$, then the curve bounds the value of p from above (Fig. 4). If $2n > b > n$, the curve bounds the value of p from below (Fig. 3). If $b = n$ (Fig. 2), then it follows from (16) that $\theta(\eta_i)$ is positive only when approaching from the rear hemisphere.

Fig. 4. Stability boundary in the case $b = 4,\ n = 5$. Each branch of the curve determines the stability boundary with respect to the rudder angular velocity for the given η_i

The investigation of the real motion and the choice of gain coefficients in the automatic motion-control system may be carried out analogously ([1,4]).

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[1] A. S. Kelzon, O. V. Grigorieva, *DAN*, **121**, No. 3 (1958).
[2] A. Locke, *Guidance*, N. Y., 1955.
[3] W. R. Ashby, *An Introduction to Cybernetics*, London, 1956.
[4] O. V. Grigorieva, A. S. Kelzon, *Scientific Notes of the Leningrad Higher Marine Engineering School named after Admiral Makarov*, No. 12 (1958).