Mathematics

1960

SovietRxiv

Abstract

Full Text

Mathematics

B. Pasynkov

On a Class of Transitive One-to-One Spectra for Bicompacta

(Presented by Academician P. S. Aleksandrov, 19 XI 1959)

In this note we solve the problem posed by P. S. Aleksandrov in 1935 of establishing a one-to-one correspondence between spectra of a certain class and bicompacta.

Definition 1. By a **spectrum** $S = \{X_\alpha, \omega_\alpha^\beta\}$ we shall mean a set of complexes X_α (the totality of indices α forms a directed partially ordered set) and simplicial mappings $\omega_\alpha^\beta : X_\beta \to X_\alpha$, defined whenever $\beta > \alpha$. The mappings $\beta > \alpha$, called **projections**, satisfy the condition of **transitivity**:

$$\omega_\alpha^\beta \omega_\beta^\gamma = \omega_\alpha^\gamma \quad \text{for } \gamma > \beta > \alpha.$$

Remark. The complexes X_α are not assumed to be complete, i.e., the set X_α need not contain, together with a simplex t_α, all its faces. If t_α is a face of t'_α, then we shall write $t_\alpha \ll t'_\alpha$. In this way X_α is turned into a partially ordered set, which may also be understood as a T_0-space ([1]).

Definition 2. A set $t = \{t_\alpha\}$, one simplex t_α from each X_α, will be called a **projection set** if, for $\beta > \alpha$, we always have

$$t_\alpha = \omega_\alpha^\beta t_\beta.$$

A projection set $t = \{t_\alpha\}$ will be called a **thread** if there is no projection set $t' = \{t'_\alpha\}$, distinct from t, such that for every α, $t_\alpha \geq t'_\alpha$. In what follows, the complexes X_α are everywhere assumed to be finite, and the projections ω_α^β are mappings "onto."

Definition 3. A simplex $t_\alpha \in X_\alpha$ will be called **minimal** if X_α contains no proper faces of t_α.

Definition 4. We shall call a simplex t_α **essential** if in some X_β ($\beta \geq \alpha$) there is a minimal simplex t_β projecting onto t_α

$$(\omega_\alpha^\beta t_\beta = t_\alpha).$$

Theorem 1. *The condition that the simplex t_α be essential is necessary and sufficient in order that "a thread pass through it" (i.e., that there exist a thread containing among its elements the simplex t_α).*

Definition 5. A spectrum S will be called **separated from above** if, for any two essential simplices t_α and t'_α, separated from below in the sense of the equality

$$[t_\alpha] \cap [t'_\alpha] = \Lambda,$$

there exists $\beta > \alpha$ for which "separation from above" takes place in the sense

$$O_\beta (\omega_\alpha^\beta)^{-1} [t_\alpha] \cap O_\beta (\omega_\alpha^\beta)^{-1} [t'_\alpha] = \Lambda,$$

where, as always, O_β denotes the star in the complex X_β, and square brackets denote closure (in the present case in the complex X_α).

Theorem 2. *The condition that the spectrum S be separated from above is equivalent to the Hausdorff condition for this spectrum, namely, that for any two threads $t = \{t_\alpha\}$ and $t' = \{t'_\alpha\}$ there exists an α such that t_α and t'_α have disjoint stars in X_α.*

By definition, the threads are the points of the **limit space** X of the spectrum S.

The topology in X is introduced as follows: a neighborhood $U_{\alpha_0} t$ of the thread $t = \{t_\alpha\}$, by definition, consists of all threads $t' = \{t'_\alpha\}$ for which $t'_{\alpha_0} \in O t_{\alpha_0}$.

If to each thread $t = \{t_\alpha\}$ we put in correspondence t_α (its α-th coordinate), then we obtain a continuous mapping (projection) $\wp_\alpha$ of the space X into X_α. It is clear that if the spectrum S is essential, i.e. all simplexes of its complexes are essential, then $\wp_\alpha$ and $\wp_\alpha^\beta$ are mappings "onto."For transitive spectra $\wp_\alpha = \wp_\alpha^\beta \wp_\beta$ for $\beta > \alpha$.

In what follows, by a spectrum is meant a Hausdorff essential spectrum.

Definition 6. We shall call a subspectrum $S' = \{X_{\alpha'}, \wp_{\alpha'}^{\beta'}\}$ of the spectrum $S = \{X_\alpha, \wp_\alpha^\beta\}$ a **nonseparating** subspectrum of the spectrum S, if: 1) in S there is an X_{α_0}, and in it such simplexes $t_{1\alpha_0}$ and $t_{2\alpha_0}$, that $[t_{1\alpha_0}] \cap [t_{2\alpha_0}] = \Lambda$; 2) for every $\beta > \alpha_0$ in X_β there exist $t_{1\beta}$ and $t_{2\beta}$, for which $\wp_{\alpha_0}^\beta t_{1\beta} = t_{1\alpha_0}$, $\wp_{\alpha_0}^\beta t_{2\beta} = t_{2\alpha_0}$, and which are "glued together" in S', i.e. $\wp_{\alpha'}^\beta t_{1\beta} \leq \wp_{\alpha'}^\beta t_{2\beta}$ for $X_{\alpha'} \in S'$. If these conditions are not fulfilled, then the subspectrum is naturally called separating.

Definition 6 can also be given the following form: a subspectrum S' of a spectrum S is separating if and only if for any two threads t_1 and t_2 of the spectrum S there exists in S' such a complex $X_{\alpha'}$ that the simplexes $t_{1\alpha'} = t_1 \cap X_{\alpha'}$ and $t_{2\alpha'} = t_2 \cap X_{\alpha'}$ have disjoint stars in $X_{\alpha'}$.

Theorem 3. *In order that a subspectrum S' of a spectrum S give the same limit space as S, it is necessary and sufficient that S' be a separating subspectrum.*

Definition 7. We shall say that a spectrum $\widetilde{S}$ is obtained from a spectrum $S = \{X_\alpha, \wp_\alpha^\beta\}$ by a **weakening of the order**, if $\widetilde{S}$ consists of the same complexes X_α as S, from $\beta > \alpha$ in $\widetilde{S}$ there follows $\beta > \alpha$ in S, and the projections in both spectra are the same.

Definition 8. Given a spectrum $S = \{X_\alpha, \wp_\alpha^\beta\}$. We shall call complexes X_α and X_β **isomorphic with respect to the spectrum** S (S-isomorphic), if between X_α and X_β one can establish an isomorphism such that, under $t_\alpha \leftrightarrow t_\beta$, the conditions $\wp_\alpha^\gamma t_\gamma = t_\alpha$ and $\wp_\beta^\gamma t_\gamma = t_\beta$ are equivalent for every $\gamma > \beta, \alpha$.

Basic definition. A spectrum Σ is called **extremal** if it satisfies the following conditions:

1) The spectrum Σ contains no two Σ-isomorphic complexes (minimality condition).
2) The spectrum Σ cannot be obtained from another spectrum by weakening the order (first maximality condition).
3) The spectrum Σ is not a separating subspectrum of any spectrum satisfying conditions 1) and 2) (second maximality condition).

Main theorem. *Every bicompactum X is the limit space of some—and, up to isomorphism, of only one—extremal spectrum.*

Remark. It is known ([1]) that every Hausdorff transitive spectrum of finite complexes, in particular an extremal one, has as its limit space a bicompactum. Therefore our main theorem establishes a one-to-one correspondence between bicompacta and their extremal spectra, i.e. solves the problem posed at the beginning of the paper.

The author expresses sincere gratitude to P. S. Aleksandrov, under whose supervision the work was carried out.

Received
18 XI 1959

CITED LITERATURE

1. P. S. **Aleksandrov**, *Uspekhi Mat. Nauk*, **2**, no. 1 (1947).

Note: Figure translations are in progress. See original paper for figures.