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MATHEMATICS

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STABILITY OF STATIONARY VALUES

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Let a group G act on a differentiable manifold E, and let a differentiable function J, invariant with respect to the group G, be defined on E (the manifold E may be infinite-dimensional, i.e., a locally Banach space, for example, a sphere in Hilbert space). By virtue of the invariance of the function J with respect to the group G, one may regard J as a function on the space of trajectories E/G and obtain stronger estimates of the number of stationary points of the function J than those which hold for an arbitrary function on the manifold E ([1]). M. A. Krasnosel' skii posed the question: can one use the invariance of the function J with respect to the group G in order to give an estimate of the number of stationary values of functions close to the function J, but no longer invariant with respect to the group G? In other words, the problem may be formulated as follows: estimate the number of stationary values of the function J that are stable under small perturbations of the function J (the perturbations need not be invariant with respect to the group G). The present note is devoted to the solution of this problem (M. A. Krasnosel' skii solved ([2]) the problem posed by him in the case when the manifold E is a sphere in Hilbert space and the group G consists of the identity transformation and the central symmetry).

In what follows we shall call a topological space a space; a continuous real-valued function, a function; a smooth manifold, a finite-dimensional closed smooth manifold; a differentiable function, a twice differentiable function; and a fiber space, a locally trivial fiber space. By $\{f \geq \alpha\}$ ($\{f < \alpha\}$) is denoted the set of points at which the function f assumes values $\geq \alpha$ ($< \alpha$).

Let f be a function defined on a space X. A number α is called a critical value of the function f if the embedding of the set $\{f < \alpha\}$ into the set $\{f \leq \alpha\}$ is not a weak homotopy equivalence (this definition differs somewhat from the usual one).

Let $p : E \to B$ be a fiber space; J a function on the base B; $\widetilde{J}$ a function on the total space E, defined by the formula $\widetilde{J} = Jp$ (the function $\widetilde{J}$ is constant on the fibers of the fibration $p : E \to B$; conversely, every function constant on the fibers can be obtained in the manner described above from a function on the base).

Theorem 1. *The set of critical values of the function J coincides with the set of critical values of the function $\widetilde{J}$.*

For the proof it suffices to refer to the following simple assertion.

Proposition 1. *If B_1 is a subset of the base B of the fibration $p: E \to B$, then the embedding of the set B_1 in B is a weak homotopy equivalence if and only if the embedding of the set $p^{-1}(B_1)$ in E is a weak homotopy equivalence.*

Corollary 1. If, under the hypotheses of Theorem 1, E is a smooth manifold and the function $\widetilde{J}$ is differentiable, then every critical value α of the function J is a stable stationary value of the function $\widetilde{J}$, provided the set $\{\widetilde{J} \leq \alpha\}$ is a neighborhood retract. (The number α is called a stable stationary value of the function $\widetilde{J}$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that every differentiable function K on the manifold E satisfying the condition $|\widetilde{J}(x) - K(x)| < \delta$ for all $x \in E$ has a stationary value differing from α by less than ε.)

Theorem 2. *Let E and B be smooth manifolds; let $p: E \to B$ be a differentiable fibration with fiber F; let J be a differentiable function on B; $\widetilde{J} = Jp$. Suppose that the function $\widetilde{J}$ has s stationary values, to which there correspond sets of stationary points of dimensions $k_1, k_2, \dots, k_s$ (we assume that the numbers k_i are arranged in decreasing order). Then the number of stable stationary values of the function $\widetilde{J}$ is bounded below by the largest number t for which*

$$k_1 + k_2 + \cdots + k_{t-1} < \operatorname{cat} B + (t-1)(\dim F - 1).$$

In the case when $\operatorname{cat} B = \dim B + 1$*, this estimate cannot be improved.*

Let M be a smooth manifold; let f be a differentiable function on M; let α be a stationary value of the function f. The set of stationary points of the function f at which the function takes the value α will be denoted by $S_\alpha(f)$. We shall say that the set of stationary points corresponding to the stationary value α of the function f stably has dimension $\geq k$, and shall write $\operatorname{s\,dim} S_\alpha(f) \geq k$, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that any function g satisfying the condition

$$|f(x) - g(x)| + |\operatorname{grad}(f(x) - g(x))| < \delta$$

for all $x \in M$, either has a stationary value differing from α by less than ε, with a set of stationary points of dimension $\geq k$, or has several stationary values differing from α by less than ε (we assume that a Riemannian metric has been introduced on M in some way, and denote by $|\operatorname{grad}(f(x) - g(x))|$ the length of the gradient vector).

Proposition 2. *If there exists a neighborhood V of the set $S_\alpha(f)$ such that for some $\varepsilon > 0$ one can find cohomology classes*

$$\xi \in H_c(\overline{V} \setminus \{f \le \alpha - \varepsilon\},\, A); \qquad \eta \in H^k(\overline{V} \setminus \{f \le \alpha - \varepsilon\},\, B),$$

such that the cohomology classes ξ and $\xi \cdot \eta$ cut out nonzero cohomology classes on the set $\overline{V} \cap \{f \le \alpha\} \setminus \{f \le \alpha - \varepsilon\}$ and zero cohomology classes on the set $(\overline{V} \cap \{f \le \alpha\} \setminus U) \setminus \{f \le \alpha - \varepsilon\}$, where U is an arbitrary neighborhood of the set $S_\alpha(f)$, then $\operatorname{s}\dim S_\alpha(f) \ge k$.

By $H(X, A)\,[H_c(X, A)]$ is denoted the cohomology group (with compact supports) in the sense of Alexander–Spanier with coefficients in the local system A; the product of cohomology classes $\xi \cdot \eta$ is regarded as an element of the group

$$H_c(\overline{V} \setminus \{f \le \alpha - \varepsilon\},\, A \otimes B).$$

In the case when the hypotheses of Proposition 2 are satisfied, we shall write $\operatorname{h}\dim S_\alpha(f) \ge k$; using this notation, Proposition 2 can be written in the form of the inequality:

$$\operatorname{s}\dim S_\alpha(f) \ge \operatorname{h}\dim S_\alpha(f).$$

Let E and B be smooth manifolds; let $p : E \to B$ be a differentiable fibration; let J be a differentiable function on B; $\widetilde{J} = Jp$. It is easy to pro-

to verify that the sets of stationary values of the functions J and $\widetilde{J}$ coincide and that

$$s\dim S_\alpha(\widetilde{J}) \le s\dim S_\alpha(J) + \dim F$$

(this follows from the relation $S_\alpha(\widetilde{J}) = p^{-1}(S_\alpha(J))$).

The question remains open as to whether the equality

$$s\dim S_\alpha(\widetilde{J}) = s\dim S_\alpha(J) + \dim F?$$

holds.

However, it can be shown that in the case when the fibration $p : E \to B$ is a regular covering, this relation holds under certain conditions. Namely, the following is true.

Theorem 3. *Let E be a smooth manifold; let G be a finite group of differentiable transformations of the manifold E, acting on E without fixed points; let $\widetilde{J}$ be a differentiable function on E invariant with respect to the group G; and let J be the corresponding function on the space of trajectories E/G. If*

$$h\dim S^*_\alpha(J) = \dim S_\alpha(J) = k,$$

then

$$h \dim S_\alpha(\widetilde{J}) = s \dim S_\alpha(\widetilde{J}) = s \dim S_\alpha(J) = k.$$

Let A be a periodic operator in the Hilbert space H, mapping the unit sphere S of the space H into itself. Suppose that the operator A satisfies the Lipschitz condition and that every power of the operator A either is the identity transformation or has only one fixed point θ. Let $F(x)$ be a weakly continuous and uniformly differentiable functional defined on the unit ball T of the space H and satisfying the conditions: a) $F(Ax) = F(x)$, $x \in S$; b) $F(x) > 0$ for $x \neq \theta$; $F(\theta) = 0$; c) $\Gamma x \neq \theta$ for $x \neq \theta$, $\Gamma\theta = \theta$ (where Γ denotes the gradient operator of the functional $F(x)$, and θ denotes the zero of the space H).

Theorem 4. *The functional $F(x)$ has an infinite number of distinct stable stationary values.*

Stability of stationary values is understood here in the same sense as in ([2]).

The existence of an infinite set of distinct stationary values for a periodic functional under different assumptions was proved by V. I. Anosov ([3]), following the scheme proposed by M. A. Krasnosel' skii ([2]).

I take this opportunity to express my gratitude to M. A. Krasnosel' skii for the interest he showed in this work.

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REFERENCES

[1] L. A. Lyusternik, *Tr. Matem. inst. im. V. A. Steklova*, **19** (1947). [2] M. A. Krasnosel' skii, *Topological methods in the theory of nonlinear integral equations*, 1956. [3] V. I. Anosov, *DAN*, **131**, No. 2 (1960).

Note: Figure translations are in progress. See original paper for figures.