THE STRUCTURE OF NONRELATIVISTIC COUNTERTERMS

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Abstract

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THE STRUCTURE OF NONRELATIVISTIC COUNTERTERMS

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For a correct definition of the product of singular Green's functions it is necessary first to regularize them, i.e., to replace them by sufficiently regular functions such that, first, they can be multiplied in the ordinary way and, second, in such a product one can in some sense pass to the limit, declaring it to be the product of the corresponding Green's functions. A theory regularized in this way cannot satisfy all physical requirements simultaneously before the regularization is removed. This circumstance, while not affecting the final results, entails an essential ambiguity in the choice of the regularization method. Leaving aside the question of the general conditions that any regularization process must satisfy, let us note that some particular choice of such conditions often proves expedient. The decisive role here belongs to considerations connected with the concrete specificity of the problem. As a well-known example one may cite quantum electrodynamics, where preference, for one reason or another, is given to gradient and relativistically invariant regularization methods. It is clear that the formal value of a regularization device is determined, depending on the situation, by those physical properties that the theory possesses before passage to the limit.

In ([1]) a regularization device was indicated which makes it possible, before passing to the limit, to achieve unitarity of the S-matrix at the cost of abandoning relativistic covariance, and the fundamental possibility was shown of constructing counterterms that ensure restoration of the correct covariance properties of the theory after passage to the limit. In the present work a method is given for the actual construction of the counterterms introduced in ([1]), for primitively divergent graphs.

In order not to complicate the details, let us consider the simplest example of the interaction of a spinor field with a pseudoscalar real meson field

$$\mathcal{L}(x) = g : \bar{\Psi}(x)\gamma^5\Psi(x) : \varphi(x).$$

In accordance with ([1]) we put

$$\operatorname{reg} \widetilde{S}^c(p) = \exp[-rp^2]\widetilde{S}^c(p), \qquad \operatorname{reg} \widetilde{D}^c(p) = \exp[-lp^2]D^c(p).$$

It is easy to verify that the expressions for the coefficient functions of primitively divergent graphs constructed with the aid of propagators regularized according to ([1]) will, in the α-representation, contain integrals of the form

$$\int P(k_1, \dots, k_n; p) \exp\{i(ap_0^2 - a'p^2) + 2i(p^0 K^0 - \mathbf{p}\mathbf{K}')\}\, dp,$$

where $P(k_1, \dots, k_n; p)$ is a polynomial in the external momenta $k_1, \dots, k$ and in the internal momentum p; the constants a and a' satisfy the conditions:

$$a > 0, \qquad a' = a - i\delta, \qquad \delta > 0,$$

and the vector with components K^0, $\mathbf{K}'$ is a linear combination of the external momenta

$$K^0 = \sum b_i k_i^0, \qquad \mathbf{K}' = \sum b_i' \mathbf{k}_i, \qquad b_i' = b_i - i\beta_i$$

with real constants b_i and β_i.

To compute this integral, consider the following auxiliary example: one-dimensional nonrelativistic motion of a free particle of mass m. Denote the wave function of this particle by $F(t, x)$. Its time evolution is uniquely determined by the Schrödinger equation

$$i\,\frac{\partial F(t,x)}{\partial t} = -\frac{1}{2m}\,\frac{\partial^2 F(t,x)}{\partial x^2}.$$

Let at the time $t = 0$ the state of the particle be described by some function $f(x)$, concerning which we assume that it grows no faster than a polynomial as $|x| \to \infty$: $F(0, x) = f(x)$.

The wave function under consideration, $F(t, x)$, can be represented in the form

$$F(t,x) = \exp\left[-\frac{1}{2im}\, t\, \frac{d^2}{dx^2}\right] f(x).$$

On the other hand, there is an expansion into an integral

$$F(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left[-i\frac{p^2}{2m}t - ipx\right] \left\{\int_{-\infty}^{\infty} f(y)\exp(ipy)\, dy\right\} dp.$$

By virtue of the uniqueness of the determination of the wave function from its initial value, these two expressions coincide. Thus we have the identity*

$$\int_{-\infty}^{\infty} f(y)\exp\left[iy^2+iyx\right]dy=\frac{1+i}{\sqrt{2}}\sqrt{\pi}\,\exp[-ix^2]\left\{\exp\left[-\frac{1}{4i}\frac{d^2}{dz^2}\right]f(z)\right\}_{z=-x}.$$

If $f(x)$ is taken to be a polynomial of degree n, then this relation can also be obtained by induction on n.

In application to the integrals that interest us, the identity derived gives

$$\int P(k_1,\dots,k_n;p)\,\exp\left[i(ap_0^2-a'\mathbf{p}^2)+2i(p^0K^0-\mathbf{p}\mathbf{K}')\right]dp=$$

$$=\frac{\pi^2}{i}\,\frac{\exp\left[-i\left(\frac{K_0^2}{a}-\frac{\mathbf{K}'^2}{a'}\right)\right]}{\sqrt{aa'^3}}\times$$

$$\times\left\{\exp\left[-\frac{1}{4i}\left(a\frac{\partial^2}{\partial l_0^2}-a'\frac{\partial^2}{\partial\mathbf{l}^2}\right)\right]P\left(k_1,\dots,k_n;\frac{l_0}{a},\frac{\mathbf{l}}{a'}\right)\right\}_{l_0=-K^0,\ \mathbf{l}=-\mathbf{K}'}.$$

* This is a special case of the following simple identity. Suppose that $\mathcal{H}f_n(x)=E_nf_n(x)$ and $f_n(x)$ form a complete orthonormal system of functions; then

$$e^{\alpha H}F(x)=\int\Phi_\alpha(x,y)F(y)\,dy,\qquad\text{where}\qquad\Phi_\alpha(x,y)=\sum_{(n)}e^{\alpha E_n}f_n(x)f_n^*(y).$$

Let us now consider the counterterm for the second-order mass operator $\tilde{\Sigma}_2(p)$. With the aid of the last formula we find:

$$\mathrm{per}\,\tilde{\Sigma}_2(p)=-i\left(\frac{g}{4\pi}\right)^2\int_0^\infty\int_0^\infty\frac{\exp\left[-i(\alpha_1M^2+\alpha_2m^2)-\varepsilon(\alpha_1+\alpha_2)+i\left(\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}p_0^2-\frac{\alpha_1'\alpha_2'}{\alpha_1'+\alpha_2'}\mathbf{p}^2\right)\right]}{\sqrt{(\alpha_1+\alpha_2)(\alpha_1'+\alpha_2')^3}}\times$$

$$\times\left(M+\frac{\alpha_2}{\alpha_1+\alpha_2}p^0\gamma^0-\frac{\alpha_2'}{\alpha_1'+\alpha_2'}\mathbf{p}\cdot\gamma\right)d\alpha_1d\alpha_2.$$

It is immediately seen that the operator

$$\tilde{\Sigma}_2(p)=\lim_{r,l\to0}\left\{\mathrm{per}\,\tilde{\Sigma}_2(p)-\tilde{R}_{\Sigma_2}(p)\right\}$$

can be made finite and Lorentz invariant by choosing in a suitable way the polynomial $\tilde{R}_{\Sigma_2}(p)$ of the first degree, in accordance with the index of the graph under consideration. If one imposes the additional conditions

$$\tilde{\Sigma}_2(0) = 0, \qquad \left.\frac{\partial \tilde{\Sigma}_2(p)}{\partial p^\nu}\right|_{p=0} = 0,$$

then $\tilde{R}_{\Sigma_2}(p)$ may be taken in the form

$$\tilde{R}_{\Sigma_2}(p) = \text{per}\,\tilde{\Sigma}_0(0) + \sum_{(\nu)} \left.\frac{\partial\ \text{per}\,\tilde{\Sigma}_2(p)}{\partial p^\nu}\right|_{p=0} p^\nu,$$

whence, in the limit for small r and l,

$$\tilde{R}_{\Sigma_2}(p) = -i\left(\frac{g}{4\pi}\right)^2 \left\{\left(M + \frac{1}{2}p\gamma\right)\left[\ln\frac{1}{M^2(r+l)} + \frac{\mu^2}{1-\mu^2}\ln\mu^2 + 1 - \psi\left(\frac{3}{2}\right) + 2\psi(1)\right]\right.$$
$$\left. + \frac{1}{2}p\gamma\left[\frac{\mu^2}{(1-\mu^2)^2}\ln\mu^2 - \frac{1}{2}\frac{1+\mu^2}{1-\mu^2}\right] + \frac{1}{3}\frac{r-l}{r+l}\,\mathbf{p}\cdot\gamma\right\},$$

where terms vanishing for $r = l = 0$ have been omitted. Thus, the counterterm contains a finite Lorentz-noninvariant addition, which is responsible for restoring the correct relativistic properties of the operator $\tilde{\Sigma}_2(p)$ in passing to the limit $r, l \to 0$.

In an analogous way all the remaining primitively divergent graphs may be considered. We give the final expressions for the corresponding counterterms:

$$\tilde{R}_{\Pi_2}(k) = -i\left(\frac{g}{2\pi}\right)^2 \left\{\frac{1}{r} + \left(M^2 - \frac{1}{2}k^2\right)\left[\ln 2M^2 r + \psi\left(\frac{3}{2}\right) - 2\psi(1)\right] - \frac{2}{3}\mathbf{k}^2\right\},$$

$$\tilde{R}_{\Gamma_3} = -i\frac{g^3}{4\pi^2}\gamma^5\left\{\ln\frac{1}{M^2(2r+l)} + \frac{\mu^2}{1-\mu^2}\ln\mu^2 - \psi\left(\frac{3}{2}\right) - \psi(1) - \psi(2)\right\},$$

$$\tilde{R}_{\square_4} = i\pi^2 g^4\left\{2\ln\frac{1}{4M^2 r} - 2\left[\psi\left(\frac{3}{2}\right) - 2\psi(1)\right] - \frac{4}{3}\right\}.$$

Of interest is a general formulation of the R-operation based on the use of the nonrelativistic regularization method (1). Here a number of questions arise which it is proposed to consider in subsequent works.

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1. B. M. Stepanov, DAN, **108**, 1045 (1956).

Note: Figure translations are in progress. See original paper for figures.