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Abstract

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THE WI-MAN–VALIRON THEOREM FOR ENTIRE FUNCTIONS OF SEVERAL VARIABLES

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In this note we generalize the well-known Wiman–Valiron theorem ((*1*), p. 211) to the case of entire functions of several complex variables

$$u = F(z_1, z_2, \dots, z_n) = \sum_{i_1, i_2, \dots, i_n = 0}^{\infty} a_{i_1, i_2, \dots, i_n} z_1^{i_1} z_2^{i_2} \cdots z_n^{i_n}. \tag{1}$$

One generalization was found in (*2*). The results of the present note, in our opinion, give a natural analogue of the Wiman–Valiron theorem, which was not achieved in (*2*). We note that the results obtained in (*2*) overlap with our work only incompletely.

Let us define the maximum modulus of the entire transcendental function (1) on the hypersurface $S_m(R) : |z_1|^m + |z_2|^m + \dots + |z_n|^m = R^m$, $m > 0$ an arbitrary number:

$$M(R, F) = M(R) = \max_{S_m(R)} |F(z_1, z_2, \dots, z_n)|. \tag{2}$$

We reduce the series (1) to diagonal form

$$F(z_1, z_2, \dots, z_n) = \sum_{k=0}^{\infty} A_k(z_1, z_2, \dots, z_n),$$

where $A_k(z_1, z_2, \dots, z_n)$ are homogeneous functions of degree k in the variables $z_1, z_2, \dots, z_n$. Let

$$B_k = \max_{S_m(1)} |A_k(z_1, z_2, \dots, z_n)|.$$

Construct the function

$$h(z) = \sum_{n=0}^{\infty} B_n z^n, \tag{3}$$

which is, obviously, entire, and define its central index $\nu(R)$. Finally, denote by $\zeta_1, \zeta_2, \dots, \zeta_n$ the point at which the maximum in (2) is attained; $\zeta_j = r_j(R)e^{i\varphi_j(R)}$, $j = 1, 2, \dots, n$. The following holds.

Theorem 1. For any entire nonnegative i_j,

$$\lim_{R\to\infty}\left(\frac{\xi_1^{i_1}\xi_2^{i_2}\cdots\xi_n^{i_n}\dfrac{\partial^{i_1+i_2+\cdots+i_n}}{\partial z_1^{i_1}\partial z_2^{i_2}\cdots\partial z_n^{i_n}}F(\xi_1,\xi_2,\dots,\xi_n)}{\nu^{i_1+i_2+\cdots+i_n}(R)F(\xi_1,\xi_2,\dots,\xi_n)}-\left(\frac{r_1}{R}\right)^{mi_1}\left(\frac{r_2}{R}\right)^{mi_2}\cdots\left(\frac{r_n}{R}\right)^{mi_n}\right)=0, \tag{4}$$

where $F(z_1, z_2, \dots, z_n)$ is an entire transcendental function, $r_j = r_j(R)$, and where, in passing to the limit, it may be necessary to exclude a set of nonintersecting intervals of the R-axis of finite logarithmic measure.

Definition. A real function $y = y(x)$, $x_1 \leqslant x \leqslant x_2$, is called an **algebraic arc** if $y(x)$ is analytic inside the interval $[x_1, x_2]$, and at the endpoints may have only algebraic singularities.

Theorem 2. The functions $M(R)$, $r_j(R)$ and $\varphi_j(R)$, $j = 1, 2, \dots, n$, are piecewise algebraic, i.e. on every finite interval $R_1 \leqslant R \leqslant R_2$ these functions consist of a finite number of algebraic arcs.

Theorem 3.

$$\lim_{R\to\infty}\frac{RM'(R)}{\nu(R)M(R)} = 1, \tag{5}$$

where, in passing to the limit, one should possibly omit a set of nonintersecting intervals of the R-axis of finite logarithmic measure.

These theorems may be applied to the study of the growth of solutions of the form

$$z_1^{\lambda_1}z_2^{\lambda_2}\cdots z_n^{\lambda_n}F(z_1, z_2, \dots, z_n), \tag{6}$$

(where $\lambda_1, \lambda_2, \dots, \lambda_n$ are constant complex numbers, and $F(z_1, z_2, \dots, z_n)$ is an entire transcendental function) of differential equations in partial derivatives. For illustration we shall restrict ourselves to the case of linear equations and equations of the first order.

Let the differential equation be

$$H(z_1, z_2, \dots, z_n, u, z_1p_1, z_2p_2, \dots, z_{np}n) = 0, \tag{7}$$

where $p_j = \partial u / \partial z_j$, and $H(z_1, z_2, \dots, z_n, u, \eta_1, \eta_2, \dots, \eta_n)$ is a polynomial with respect to all variables and of degree q with respect to $\eta_1, \eta_2, \dots, \eta_n$, and has a solution of the form (6). Suppose further that the coefficients of all terms of degree q with respect to $\eta_1, \eta_2, \dots, \eta_n$ are constant numbers. Rewrite (7) in the following form:

$$\sum_{j_1+j_2+\dots+j_n=q} a_{j_1 j_2 \dots j_n} (z_1 p_1)^{j_1} (z_2 p_2)^{j_2} \cdots (z_{np} n)^{j_n} +$$

$$+ H_1(z_1, z_2, \dots, z_n, u, z_1 p_1, z_2 p_2, \dots, z_{np} n) = 0. \tag{8}$$

We identify the order of a solution of the form (6) with the order of growth of the function $M(R, F)$.

Theorem 4. If the homogeneous form

$$\sum_{j_1+j_2+\dots+j_n=q} a_{j_1 j_2 \dots j_n} \eta_1^{j_1} \eta_2^{j_2} \cdots \eta_n^{j_n}$$

does not vanish for nonnegative $\eta_1, \eta_2, \dots, \eta_n$ on the hypersurface S_m (1), then any solution of the form (6) of equation (8) has finite order, not exceeding a certain constant number ρ, common to all solutions (of the indicated form) of the equation under consideration.

The situation is analogous for solutions of linear equations of the form

$$\sum_{j_1+j_2+\dots+j_n=q} a_{j_1 j_2 \dots j_n} z_1^{j_1} z_2^{j_2} \cdots z_n^{j_n} \frac{\partial^q u}{\partial z_1^{j_1} \partial z_2^{j_2} \cdots \partial z_n^{j_n}} + \tag{9}$$

$$+ \sum_{l=0}^{q-1} \sum_{j_1+j_2+\dots+j_n=l} z_1^{j_1} z_2^{j_2} \cdots z_n^{j_n} P_{j_1 j_2 \dots j_n}(z_1, z_2, \dots, z_n) \frac{\partial^l u}{\partial z_1^{j_1} \partial z_2^{j_2} \cdots \partial z_n^{j_n}} = 0,$$

where $P_{j_1 j_2 \dots j_n}(z_1, z_2, \dots, z_n)$ are polynomials and $a_{j_1 j_2 \dots j_n}$ are constant numbers.

Theorem 5. Equation (9) has an infinite set of solutions of the form (6), provided that the homogeneous form

$$\sum_{j_1+j_2+\dots+j_n=q} a_{j_1 j_2 \dots j_n} \eta_1^{j_1} \eta_2^{j_2} \cdots \eta_n^{j_n}$$

does not vanish for nonnegative $\eta_1, \eta_2, \dots, \eta_n$ on the hypersurface $S_1(1)$.

This theorem generalizes the well-known theorem of Fuchs on regular singularities of solutions of ordinary differential equations (([3], pp. 214–221).

Theorem 6. If the homogeneous form

$$\sum_{j_1+j_2+\cdots+j_n+q} a_{j_1 j_2 \cdots j_n} \eta_1^{j_1} \eta_2^{j_2} \cdots \eta_n^{j_n}$$

for nonnegative $\eta_1, \eta_2, \dots, \eta_n$ does not vanish on the hypersurface $S_m(1)$, then every solution of the form (6) of equation (9) has finite order not exceeding a certain constant number ρ, common to all solutions (of the indicated form) of the equation under consideration.

In all three of the last theorems, the condition that the homogeneous form not vanish is essential, since if this condition is not satisfied, one can construct equations of the types studied that possess solutions of infinite order of the form (6), or that have no solutions at all of the types studied.

Let us note that the number m in the definition of the hypersurface $S_m(R)$ may be chosen arbitrarily. One may, for example, take $m = 1$ (a hypercone) or $m = 2$ (a hypersphere). Finally, we also point out the circumstance that the Wiman–Valiron theorem can be generalized also to the case where, instead of the hypersurfaces $S_m(R)$, one takes hypersurfaces of a more general form; however, in doing so nothing essentially new is obtained.

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Note: Figure translations are in progress. See original paper for figures.