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Abstract

Full Text

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DYNAMICAL SYSTEMS WITH COMPLETELY POSITIVE AND ZERO ENTROPY*

(Presented by Academician A. N. Kolmogorov on 6 IV 1960)

Let in a Lebesgue space M with Boolean algebra α of measurable subsets of M and measure $\mu(\cdot)$, defined on α, there be given a dynamical system $\{S_t\}$, i.e., a one-parameter group of automorphisms of the space M (see (1)). If ξ is a measurable partition of M, invariant with respect to $\{S_t\}$, then in the factor space $M|\xi$ there is induced a dynamical system $\{S'_t\}$, called a factor-system of the system $\{S_t\}$.

Definition 1. A dynamical system $\{S_t\}$ is called a system with **completely positive entropy** if every nontrivial factor-system has positive entropy (for the entropy of dynamical systems see (2–4)).

Definition 2. A dynamical system $\{S_t\}$ is called **regular in the sense of A. N. Kolmogorov** (2), if there exists a closed subalgebra α_0 of the algebra α whose shifts $\alpha_t = S_t\alpha_0$ have the following properties:

$$\alpha_t \subseteq \alpha_{t'} \quad \text{for } t \leq t'; \tag{1}$$

$$\bigvee_t \alpha_t = \alpha; \tag{2}$$

$$\bigwedge_t \alpha_t = \mathfrak{N}; \tag{3}$$

$\mathfrak{N}$ is the trivial subalgebra of the Boolean algebra α.

Definition 3. A dynamical system $\{S_t\}$ is called **singular in the sense of Kolmogorov** if every closed subalgebra α_0 satisfying conditions (1) and (2) coincides with α.

Definition 4. A closed subalgebra α_0 of the algebra α is called a **generating** one for the dynamical system $\{S_t\}$ if

$$\bigvee_t \alpha_t = \alpha.$$

Definition 5. Factor-systems $\{S'_t\}$ and $\{S''_t\}$ of the dynamical system $\{S_t\}$, defined on the factor spaces $M|\xi'$ and $M|\xi''$, respectively, are called **mutually independent** if, for any $K' \in \xi'$, $K'' \in \xi''$,** one has

$$\mu(K' \cdot K'') = \mu(K')\mu(K'').$$

We introduce notation. For finite subalgebras β_0 of the algebra α and $h > 0$, set

$$\beta^{(h)} = \bigvee_n S_{nh}\beta_0, \qquad \bar{\beta}^{(h)} = \bigwedge_n \bigvee_{m<n} S_{mh}\beta_0$$

and

$$\bar{\alpha} = \bigvee_{\substack{h>0 \\ \beta_0 \subseteq \alpha}} \bar{\beta}^{(h)},$$

and let $\{\bar{S}_t\}$ be the factor-system of the system $\{S_t\}$ acting in the factor space $M|\bar{\alpha}$; $\{S_n(\beta^{(h)})\}$ and $\{S_n(\bar{\beta}^{(h)})\}$ are factor-systems of the system $\{S_{nh}\}$, generated by auto-

* The terminology used in the present note was proposed by A. N. Kolmogorov, V. A. Rokhlin, and Ya. G. Sinai.
** We denote a partition of the space M and the σ-algebra generated by this partition by the same Greek letter.

by the morphism S_h, acting in the factor spaces $M \mid \beta^{(h)}$ and $M \mid \bar{\beta}^{(h)}$, respectively.

Theorem 1. Any factor system of a dynamical system $\{S_t\}$ with completely positive or zero entropy is a system with completely positive or zero entropy, respectively.

The proof of this theorem follows directly from the corresponding definitions.

Theorem 2. In order that $\{S_t\}$ be a dynamical system with completely positive or zero entropy, the following condition is necessary and sufficient.

For any finite subalgebra β_0 and any admissible $h > 0$, the system $\{S_n(\beta^{(h)})\}$ is, respectively, regular or singular in the sense of Kolmogorov.

Theorem 3. A dynamical system regular in the sense of Kolmogorov has completely positive entropy; a dynamical system with zero entropy is singular in the sense of Kolmogorov.

Corollary. A dynamical system $\{S_t\}$ with a finite generator is regular or singular in the sense of Kolmogorov if and only if $\{S_t\}$ is a system with completely positive or zero entropy, respectively.

Theorem 4. The factor system $\{\bar{S}_t\}$ has zero entropy. Any factor system with zero entropy is a factor system of $\{\bar{S}_t\}$.

Theorem 5. The factor systems $\{S'_t\}$ and $\{\bar{S}_t\}$, with completely positive and zero entropy, respectively, are mutually independent.

Here the following questions arise: do maximal factor systems with completely positive entropy exist for every dynamical system, and can every ergodic dynamical system be decomposed into a direct product of independent factor systems with completely positive or zero entropy?

Theorem 6. A dynamical system generated by a multidimensional stationary Gaussian random process with absolutely continuous spectral functions is a system with completely positive entropy. A dynamical system generated by a multidimensional stationary random process with singular spectral functions has zero entropy.

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CITED LITERATURE

1. V. A. Rokhlin, *UMN*, **4**, No. 2 (30), 57 (1949).

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3. Ya. G. Sinai, *DAN*, **124**, No. 4, 768 (1959).

4. A. M. Abramov, *DAN*, **128**, No. 5, 873 (1959).

Note: Figure translations are in progress. See original paper for figures.