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Abstract

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ON THE SUMMABILITY OF SERIES IN PRINCIPAL VECTORS OF NON-SELF-ADJOINT OPERATORS

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Let C be a linear completely continuous operator acting in a Hilbert space $\mathfrak{H}$. Let $\lambda_1, \lambda_2, \ldots, \lambda_s, \ldots$ be the characteristic numbers* of the operator C, arranged in order of increasing modulus, and let

$$\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_s, \ldots \tag{1}$$

be the system of corresponding eigenvectors and associated vectors (hereinafter principal vectors) of the operator C.

It can be shown that the system (1) always admits a biorthogonal system

$$\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_s, \ldots, \tag{2}$$

composed of principal vectors of the adjoint operator C^*. In view of this, if for some $f \in \mathfrak{H}$ the equality

$$\mathbf{f} = \sum_{s=1}^{\infty} c_s \mathbf{e}_s, \tag{3}$$

holds, where the series on the right converges in the metric of the Hilbert space, then the coefficients c_s are determined in a natural way (cf. (^{1})). Namely, as can be shown, if $\mathbf{e}_s$ is an eigenvector of C having no associated vectors (hereinafter a simple eigenvector), then

$$c_s = \frac{(\mathbf{f}, \mathbf{g}_s)}{(\mathbf{e}_s, \mathbf{g}_s)}, \tag{4}$$

where $\mathbf{g}_s$ is an eigenvector of C^* corresponding to the characteristic number $\overline{\lambda}_s$. If, however, the vectors $\mathbf{e}_p, \mathbf{e}_{p+1}, \ldots, \mathbf{e}_{p+k}$ form a Jordan chain corresponding to the characteristic number λ_p, then

$$c_{p+i} = \frac{(\mathbf{f}, \mathbf{g}_{p+k-i})}{(\mathbf{e}_{p+i}, \mathbf{g}_{p+k-i})} \quad (0 \le i \le k). \tag{5}$$

Here $\mathbf{g}_p, \mathbf{g}_{p+1}, \dots, \mathbf{g}_{p+k}$ is a Jordan chain of vectors of the operator C^*, corresponding to the characteristic number $\overline{\lambda}_p$. In view of this observation, to every $f \in \mathfrak{H}$ one can associate the formal Fourier series in the principal vectors of the operator C

$$\mathbf{f} \sim \sum_{s=1}^{\infty} c_s \mathbf{e}_s, \tag{6}$$

where the coefficients c_s are determined by formulas (4) and (5).

* $\lambda_s = 1/\mu_s$, where μ_s are the nonzero eigenvalues of the operator C.

At the present time, broad conditions have been found under which the system of principal vectors (1) is complete (1-5). However, the Fourier series (6) in this case, generally speaking, diverge. In the present paper it is proved that the Fourier series in the principal vectors of an operator are summable to the corresponding element **f** by Abel's method. Thus, among other things, a method is given that makes it possible to find the coefficients of linear combinations of elements of the system (1) approximating **f** with a prescribed accuracy in advance. It is also established, in passing, that the solution of the Cauchy problem for the equation $\partial u/\partial t + Bu = 0$ is expanded, under certain assumptions concerning the operator B, into a convergent Fourier series in the principal vectors of the operator.

For a precise formulation of the results we shall need some definitions.

Having fixed some $\alpha > 0$, we introduce for consideration the following polynomials in t:

$$P_m^{\alpha}(\zeta^{-1}, t) = \frac{e^{\zeta^{-\alpha} t}}{m!} \frac{d^m}{d\zeta^m} e^{-\zeta^{-\alpha} t}, \qquad (m = 0, 1, \dots), \tag{7}$$

after which we associate with the series (6) the series

$$\sum_{s=1}^{\infty} c_s(t) \mathbf{e}_s. \tag{8}$$

The coefficients $c_s(t)$ of this series are formed from the coefficients c_s of the series (6) and the characteristic numbers λ_s in the following way. If $\mathbf{e}_s$ is a simple eigenvector, then

$$c_s(t) = e^{-\lambda_s^{\alpha} t} c_s. \tag{9}$$

If the vectors $\mathbf{e}_p, \mathbf{e}_{p+1}, \dots, \mathbf{e}_{p+k}$ form a Jordan chain, then

$$c_{p+i}(t) = e^{-\lambda_p^\alpha t} \sum_{m=0}^{k-i} P_m^\alpha(\lambda_p, t) c_{p+i+m}. \tag{10}$$

We note that, as $t \to 0$, in all cases $c_s(t) \to c_s$.

Definition. Suppose the series (8) has a subsequence of partial sums $\{S_{N_\nu}(t)\}$ which converges for all $t > 0$, and let $\mathbf{u}(t)$ be the corresponding limiting function

$$\mathbf{u}(t) = \sum_{\nu=1}^{\infty} \left(\sum_{s=N_\nu+1}^{N_{\nu+1}} c_s(t)\mathbf{e}_s \right). \tag{11}$$

Suppose, moreover, that

$$\lim_{t \to +0} \mathbf{u}(t) = \mathbf{f}. \tag{12}$$

Then we shall say that the Fourier series (6) corresponding to the element $\mathbf{f}$ is summable to $\mathbf{f}$ by the method (A, λ, α).

Theorem 1. Let C be a linear completely continuous operator in $\mathfrak{H}$. Suppose that, for some $\rho > 0$, the series converges

$$\sum_{s=1}^{\infty} r_s^\rho, \tag{13}$$

where γ_s are the eigenvalues of the operator $(C^*C)^{1/2}$, and suppose that the values of the quadratic form (Ch, h) lie in the sector*

$$-\frac{\pi}{2\rho'} \leq \arg z \leq \frac{\pi}{2\rho'}, \qquad \rho' > \max\left(\rho, \frac{1}{2}\right). \tag{14}$$

Then the Fourier series of any element f belonging to the range of the operator C, i.e. representable in the form $f = Ch$, is summable to f by the method (A, λ, α) for all α satisfying the condition $\rho' > \alpha > \rho$, and in the case of integral ρ the condition $\rho' > \alpha \geq \rho$.

For the proof we consider the integral

$$\mathbf{u}(t) = \frac{1}{2\pi i} \int_\gamma e^{-\lambda^\alpha t} (E - \lambda C)^{-1} C f \, d\lambda. \tag{15}$$

This integral is taken over the boundary of the domain G, which is the sector $-\pi/2\rho' - \varepsilon < \arg z < \pi/2\rho' + \varepsilon$ ($\varepsilon > 0$ and sufficiently small), from which a neighborhood of the origin has been removed, so small that in it the function $R_\lambda f = (E - \lambda C)^{-1} f$ is regular. It can be shown that on the contour γ the resolvent R_λ is bounded; therefore, for $t > 0$ the integral (15) exists. The further proof is based on computing the integral (15) by means of residue theory. One considers a system of contours Γ_ν receding to infinity, whose ends slide along the lower and upper rays of the contour γ. In the part of the domain G cut off by the contour Γ_ν there are N_ν poles of the resolvent R_λ. What is essential is that the contours Γ_ν can be chosen so that on them the inequality

$$\|R_\lambda f\| \leq e^{o(|\lambda|^\alpha)} \|f\| \tag{16}$$

is satisfied. Since, by virtue of this estimate, for fixed $t > 0$

$$\int_{\Gamma_\nu} e^{-\lambda^\alpha t} R_\lambda f \, d\lambda \to 0,$$

formula (11) already follows from this. The existence of contours Γ_ν on which the inequality (16) is valid is established by representing the resolvent in the form $R_\lambda f = D_C(\lambda)/\Delta(\lambda)$, where $D_C(\lambda)$ and $\Delta(\lambda)$ are respectively the first minor and the Fredholm determinant of the operator C. As was first proved in full generality by M. V. Keldysh (^{6}), under condition (13) $D_C(\lambda)$ and $\Delta(\lambda)$ are entire functions of order not exceeding ρ. This assertion can be supplemented in the sense that, in the case of integral ρ, both functions $D_C(\lambda)$ and $\Delta(\lambda)$ are of minimal type. Relying on these facts and using known lower estimates for the modulus of an entire function (see (^{7}), p. 33), we obtain the inequality (16). The fact that $\mathbf{u}(t) \to f$ as $t \to +0$ is proved without difficulty.

Let us make a few further remarks.

In the case $\alpha > \rho$ the sequence of contours Γ_ν can be chosen very dense. In formula (11), only those elements of the system (1) are combined which correspond either to one and the same eigenvalue, or to different but "exponentially close" eigenvalues. It should be noted that, as examples show, complete separation cannot be achieved even in the case of a simple spectrum.

We note that for $\alpha = 1$ the function $\mathbf{u}(t)$ in (15), under the condition $C = B^{-1}$, is a solution of the Cauchy problem for the equation $\partial u/\partial t + Bu = 0$.

From our arguments there therefore follows the following fact, which we formulate for the case of differential operators:

Theorem 2. Let L be a strongly elliptic operator of order $2m$, acting in a Hilbert space of functions defined

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* We note that the values of a quadratic form always fill a convex set; therefore, in the case $\rho' < 1$, condition (14) means that the values (Ch, h) do not go outside a certain half-plane contained in the indicated sector.

in a certain finite domain D of an n-dimensional space with smooth boundary S. Let $2m > n$.

Then the solution $u(t)$ of the mixed problem

$$\frac{\partial u}{\partial t} + Lu = 0,$$

$$u\big|_{t=0} = f, \qquad u\big|_S = \frac{\partial u}{\partial \nu}\bigg|_S = \dots = \frac{\partial^m u}{\partial \nu^m}\bigg|_S = 0$$

is expanded, for all $t > 0$, into a convergent Fourier series with respect to the principal vectors of the operator L:

$$u(t) = \sum_{\nu=1}^{\infty} \sum_{s=N_\nu+1}^{N_{\nu+1}} \omega_s(t) e_s.$$

Let us explain that the series (13) corresponding to the resolvent of a strongly elliptic operator converges for all $\rho > n/2m$. Since, moreover, the values of the quadratic form $((L + \mu^2 E)^{-1} h, h)$ lie in a certain sector of the right half-plane, we have $\rho' > 1$, and under the condition $n/2m < 1$ one may set $\alpha = 1$.

We point out, in conclusion, that Theorem 2 does not cover the case of an equation of the second order

$$\frac{\partial u}{\partial t} = \sum_{p,q=1}^{2} a_{pq}(x_1, x_2) \frac{\partial^2 u}{\partial x_p \partial x_q} + \sum_{s=1}^{2} b_s(x_1, x_2) \frac{\partial u}{\partial x_s} + c(x_1, x_2) u, \qquad (17)$$

where the right-hand side is an elliptic operator (here $2m = n$, $\rho > 1$, and therefore one cannot set $\alpha = 1$). Nevertheless, we have succeeded in showing that also in this case the solution $u(t)$ is expanded, for all $t > 0$, into a convergent Fourier series with respect to the principal vectors. In the proof we used the relative smallness of the non-self-adjoint part of the elliptic operator (17) and, for estimating the resolvent in the integral (15), invoked the results of T. Carleman [(8)] and M. S. Livshits [(9)].

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Note: Figure translations are in progress. See original paper for figures.