MATHEMATICS

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1960

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Abstract

Full Text

MATHEMATICS

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ON A COMPARISON PRINCIPLE FOR SECOND-ORDER DIFFERENTIAL EQUATIONS

(Presented by Academician I. G. Petrovskii, 22 VI 1960)

1. Let two equations be given

$$x_1'' + \varphi_1(t)x_1 = 0, \tag{1}$$

$$x_2'' + \varphi_2(t)x_2 = 0, \tag{2}$$

where the coefficients $\varphi_1(t)$, $\varphi_2(t)$ are assumed summable on the interval $[a,b]$ (here equations (1), (2), as well as all differential equations encountered below, are to be understood as being satisfied almost everywhere on $[a,b]$). Speaking of solutions of equations (1), (2), we shall everywhere in what follows have in mind nontrivial solutions.

2. **Theorem 1.** *Let $x_1(t)$ not vanish on $[a,b]$. Then:*

1) *if on $[a,b]$ the inequality*

$$-\frac{x_1'(a)}{x_1(a)} + \int_a^t \varphi_1(\tau)\,d\tau > \left|-\frac{x_2'(a)}{x_2(a)} + \int_a^t \varphi_2(\tau)\,d\tau\right|, \qquad a \le t \le b, \tag{3}$$

holds, then $x_2(t)$ does not vanish on $[a,b]$ and

$$-\frac{x_1'(t)}{x_1(t)} > \left|\frac{x_2'(t)}{x_2(t)}\right|, \qquad a \le t \le b; \tag{4}$$

2) *if on $[a,b]$ the inequality*

$$\frac{x_1'(b)}{x_1(b)} + \int_t^b \varphi_1(\tau)\,d\tau > \left|\frac{x_2'(b)}{x_2(b)} + \int_t^b \varphi_2(\tau)\,d\tau\right|, \qquad a \le t \le b, \tag{5}$$

holds, then $x_2(t)$ does not vanish on $[a,b]$ and

$$\frac{x_1'(t)}{x_1(t)} > \left|\frac{x_2'(t)}{x_2(t)}\right|, \qquad a \le t \le b. \tag{6}$$

If (3) (respectively, (5)) is fulfilled with the sign of a non-strict inequality, then (4) (respectively, (6)) is also valid with the sign of a non-strict inequality.

Proof. Assertion 2) follows from 1) with the aid of the change $\tilde{x}_i(t) = x_i(a + b - t)$, $i = 1, 2$. It is therefore sufficient to prove assertion 1). The substitution

$$z_i(t) = -\frac{x_i'(t)}{x_i(t)}, \qquad i = 1, 2, \tag{7}$$

transforms, as is known, equations (1), (2) into the Riccati equations

$$z_1' = z_1^2 + \varphi_1(t), \tag{8}$$

$$z_2' = z_2^2 + \varphi_2(t). \tag{9}$$

The functions $z_i(t)$ are, evidently, continuous everywhere except at the zeros of $x_i(t)$, which are points of infinite discontinuity for the corresponding $z_i(t)$. Since $x_1(t)$ does not vanish on $[a, b]$, $z_1(t)$ is continuous on $[a, b]$ and, consequently, (8) is equivalent to the integral equation

$$z_1(t) = z_1(a) + \int_a^t z_1^2(\tau)\, d\tau + \int_a^t \varphi_1(\tau)\, d\tau, \tag{10}$$

whence, taking (3) into account, in particular it follows that

$$z_1(t) \geq z_1(a) + \int_a^t \varphi_1(\tau)\, d\tau > 0, \qquad a \leq t \leq b. \tag{11}$$

Let us now turn to the behavior of the function $z_2(t)$ on the interval $[a, b]$. From condition (3) it follows (for $t = a$) that

$$|z_2(a)| < z_1(a). \tag{12}$$

Therefore $z_2(t)$ is continuous at the point $t = a$, and hence also on some interval $[a, c]$, $a < c \leq b$. On this interval (9) can be rewritten in the form of the integral equation

$$z_2(t) = z_2(a) + \int_a^t z_2^2(\tau)\, d\tau + \int_a^t \varphi_2(\tau)\, d\tau, \qquad a \leq t \leq c. \tag{13}$$

We shall prove that on $[a, c]$ the inequality

$$|z_2(t)| < z_1(t), \qquad a \leq t \leq c \tag{14}$$

holds. Indeed, suppose the contrary. Then two cases may occur:

a) For some $t = t_0$, $a < t_0 \leq c$,

$$z_2(t_0) \leq -z_1(t_0). \tag{15}$$

b) For some $t = t_0$, $a < t_0 \leq c$,

$$z_2(t_0) \geq z_1(t_0). \tag{16}$$

We show that neither of these cases can occur:

a) Taking (13), (3), and (11) into account, we find that for any t_0 in $[a, c]$

$$z_2(t_0) = z_2(a) + \int_a^{t_0} z_2^2(\tau)\, d\tau + \int_a^{t_0} \varphi_2(\tau)\, d\tau \geq z_2(a) + \int_a^{t_0} \varphi_2(\tau)\, d\tau >$$

$$> -z_1(a) - \int_a^{t_0} \varphi_1(\tau)\, d\tau \geq -z_1(t_0). \tag{17}$$

b) Suppose (16) holds. Then, by virtue of the continuity of $z_1(t)$, $z_2(t)$ on $[a, c]$ and relation (12), there exists a point $t = t_1$, $a < t_1 \leq t_0$, such that

$$z_2(t_1) = z_1(t_1); \tag{18}$$

$$z_2(t) < z_1(t) \quad \text{for } a \leq t < t_1. \tag{19}$$

Relations (17) and (19) show that

$$|z_2(t)| < z_1(t), \qquad a \leq t < t_1,$$

whence

$$\int_a^{t_1} z_2^2(\tau)\, d\tau < \int_a^{t_1} z_1^2(\tau)\, d\tau. \tag{20}$$

Taking (13), (3), (20), and (10) into account, we obtain

$$z_2(t_1) = z_2(a) + \int_a^{t_1} \varphi_2(\tau)\, d\tau + \int_a^{t_1} z_2^2(\tau)\, d\tau <$$

$$< z_1(a) + \int_a^{t_1} \varphi_1(\tau)\, d\tau + \int_a^{t_1} z_1^2(\tau)\, d\tau = z_1(t_1),$$

which contradicts relation (18).

We have thus shown that (14) is fulfilled on any interval $[a, c] \subset [a, b]$ of continuity of $z_2(t)$. But, since $z_1(t)$ is bounded on $[a, b]$, while $z_2(t)$ can have only infinite discontinuities, it follows that $z_2(t)$ is continuous everywhere on $[a, b]$. Therefore inequality (14) holds on the whole segment $[a, b]$. Assertion 1) is proved.

We have proved Theorem 1 in the part concerning strict inequalities. The case of non-strict inequalities is obtained from this by a limiting passage, whose validity is due to the uniqueness of the solution of the Riccati equation passing through a given point.

Remark 1. With respect to the signs of the inequalities, the theorem can be sharpened in the following way. If (3) (respectively (5)) is fulfilled on $[a, b]$ with the sign $\geq$, but for some $t = t_0$, $a \leq t_0 \leq b$, the sign $>$ holds, then (4) (respectively (6)) is also fulfilled with the sign $\geq$, and for all t from the interval $[t_0, b]$ (respectively $[a, t_0]$) the sign $>$ holds.

Remark 2. The requirements (3), (5) can be weakened if additional characteristics of the functions $\varphi_1(t), \varphi_2(t)$, connected with repeated integration, are brought into consideration. For example, (3) can be weakened if, in addition to the functions

$$\Phi_i(t) = -\frac{x_i'(a)}{x_i(a)} + \int_a^t \varphi_i(\tau) d\tau, \qquad i = 1, 2, \tag{21}$$

one introduces into consideration the functions

$$\psi_i(t) = \int_a^t \Phi_i^2(\tau) d\tau, \qquad i = 1, 2.$$

Because of lack of space we shall not dwell on this.

3. Corollary. In case 1), on $(a, b]$ the inequality

$$\frac{x_2'(t)}{x_2(t)} - \frac{x_1'(t)}{x_1(t)} > \Phi_1(t) - \Phi_2(t), \qquad a < t \leq b,$$

holds, where $\Phi_i(t)$ are given by formula (21). For the proof it suffices to compare (4), (10), and (13).

Similarly, in case 2), on $[a, b)$ the inequality

$$\frac{x_1'(t)}{x_1(t)} - \frac{x_2'(t)}{x_2(t)} > \frac{x_1'(b)}{x_1(b)} - \frac{x_2'(b)}{x_2(b)} + \int_t^b [\varphi_1(\tau) - \varphi_2(\tau)]\, d\tau, \qquad a \leq t < b.$$

holds.

4. Introduce the notation

$$\int_{(\alpha,\beta)} f(\tau)d\tau = \int_{\min\{\alpha,\beta\}}^{\max\{\alpha,\beta\}} f(\tau)d\tau.$$

It follows from Theorem 1 that:

Theorem 2. Let a solution $x_2(t)$ of equation (2) satisfy the conditions

$$x_2(a) = x_2(b) = x_2'(c) = 0, \qquad a < c < b. \tag{22}$$

Further, suppose that for the coefficients of equations (1), (2) on $[a, b]$ the inequality

$$\int_{(c,t)} \varphi_1(\tau)d\tau \geq \left| \int_{(c,t)} \varphi_2(\tau)d\tau \right|, \qquad a \leq t \leq b. \tag{23}$$

is fulfilled. Then every solution of equation (1) has at least one zero in the interval $[a, b]$.

Proof. Consider some nonzero solution $x_1(t)$ of equation (1) satisfying the condition $x_1'(c) = 0$. It is easy to see that $x_1(t)$ has at least one zero in each of the half-intervals $[a, c)$ and $(c, b]$. Indeed, if $x_1(t)$ had no zeros, for example, in $(c, b]$, then we would be in the conditions of part 1) of Theorem 1, whence it would follow that $x_2(t)$ has no zeros in $(c, b]$. But this is certainly false, since $x_2(b) = 0$. Similarly, using part 2) of Theorem 1, it is established that $x_1(t)$ vanishes in $[a, c)$.

Thus $x_1(t)$ has at least two zeros on $[a, b]$. But, by the zero-separation theorem, in this case every solution of equation (1) has at least one zero on $[a, b]$. The theorem is proved.

5. The results obtained above make it possible to establish comparison theorems for second-order equations in self-adjoint form:

$$\frac{d}{dt}\left(k_1(t)\frac{dx_1}{dt}\right) + q_1(t)x_1 = 0, \tag{24}$$

$$\frac{d}{dt}\left(k_2(t)\frac{dx_2}{dt}\right) + q_2(t)x_2 = 0 \tag{25}$$

(as usual, it is assumed that $k_1(t)$, $k_2(t) > 0$).

Here we shall confine ourselves to the formulation of the following result:

Theorem 3. Let $q_1(t), q_2(t) \geq 0$. Suppose, further, that a nonzero solution $x_2(t)$ of equation (25) satisfies conditions (22), with a and b being adjacent zeros.

In order that every solution of equation (24) vanish on $[a, b]$, it is sufficient that the following conditions be fulfilled:

$$\int_a^t \frac{d\tau}{k_1(\tau)} \geq \int_a^t \frac{d\tau}{k_2(\tau)}, \qquad \int_t^c q_1(\tau)d\tau \geq \int_t^c q_2(\tau)d\tau \qquad \text{for } a \leq t \leq c;$$

$$\int_t^b \frac{d\tau}{k_1(\tau)} \geq \int_t^b \frac{d\tau}{k_2(\tau)}, \qquad \int_c^t q_1(\tau)d\tau \geq \int_c^t q_2(\tau)d\tau \qquad \text{for } c \leq t \leq b.$$

Theorem 3 may be regarded as a strengthening of the classical Sturm comparison theorem for nonnegative $q_1(t), q_2(t)$.

I express my sincere gratitude to my supervisor M. A. Krasnosel' skii.

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Received
18 VI 1960

Note: Figure translations are in progress. See original paper for figures.

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