ON n-DIMENSIONAL WIDTHS OF CERTAIN FUNCTIONAL CLASSES

1960

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Abstract

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MATHEMATICS

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ON n-DIMENSIONAL WIDTHS OF CERTAIN FUNCTIONAL CLASSES

(Presented by Academician A. N. Kolmogorov on 14 IX 1959)

1. Let F be a set lying in a Banach space R, consisting of elements $\{x, y, \dots, f, \dots\}$; let L_n be some linear n-dimensional subspace. We introduce the quantities

$$\rho(x, L_n) = \inf_{y \in L_n} \|x - y\|, \qquad \delta(F, L_n) = \sup_{x \in F} \rho(x, L_n).$$

The latter quantity will be called the **deviation of the set F from the subspace L_n**. The lower bound of $\delta(F, L_n)$ over all subspaces L_n of dimension n will be called the **n-dimensional width of the set** F and will be denoted by $d_n(F)$. n-Dimensional widths for functional classes were introduced by A. N. Kolmogorov in the paper ([1]), where they were computed for certain classes of functions in the metric L^2.

We prove the following theorem on n-dimensional widths*.

Theorem. *Let U be the unit sphere in the space R; let F_{n+1} and U_{n+1} be sections of the sets F and U by some $(n+1)$-dimensional subspace L_{n+1}. Then, if $\alpha U_{n+1} \subseteq F_{n+1}$, then $d_n(F) \geq \alpha$.*

Using this theorem, we compute two n-dimensional widths in the metric C of the following classes, known in approximation theory, of 2π-periodic functions with mean value zero:

a) the class F_r of real functions $f(x)$ for which

$$\operatorname{vrai\,max} |f^{(r)}(x)| \leq 1;$$

b) the class A_h of functions $f(z) = f(x + iy)$, real on the real axis, regular in the strip $-h < y < +h$ and satisfying there the inequality $|\operatorname{Re} f(z)| < 1$;

c) the class Γ_ρ, $\rho < 1$, of functions $f(\theta) = u(\rho, \theta)$, $0 \leq \theta \leq 2\pi$, where the functions $u(r, \theta)$, $0 \leq r \leq 1$, $0 < \theta \leq 2\pi$, are harmonic in the disk of radius one and satisfy in it the inequality $|u(r, \theta)| < 1$,

and also

d) the class B_k of functions $f(z)$, analytic in the disk $|z| < 1$, for which $|f^{(k)}(z)| < 1,\ |z| < 1$.

We shall show that:

$$d_{2n}(F_r) = \delta(F_r, T_{2n}) = \frac{4}{\pi(n+1)^r} \sum_{m=0}^{\infty} \frac{(-1)^m (r+1)}{(2m+1)^r + 1}; \tag{I}$$

$$d_{2n}(A_h) = \delta(A_h, T_{2n}) = \frac{4}{\pi} \sum_{m=0}^{\infty} (-1)^m \frac{1}{(2m+1)} \frac{1}{\operatorname{ch}(2m+1)(n+1)h}; \tag{II}$$

$$d_{2n}(\Gamma_\rho) = \delta(\Gamma_\rho, T_{2n}) = \frac{4}{\pi} \operatorname{arc\,tg} \rho^{n+1}. \tag{III}$$

* The expression $x+\gamma U$, where $\gamma > 0$, U is a set, denotes here the set of points representable in the form $x+$

The values of the deviations for the listed classes from the space T_{2n} of polynomials

$$\sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$$

were computed long ago in the works of J. Favard ([2]) and N. I. Akhiezer and M. G. Krein ([3]) for the class F_r, of N. I. Akhiezer ([4]) for the class A_h, and of M. G. Krein ([5]) for the class Γ_ρ.

Relations (I)–(III) show that the subspace T_{2n} is "closest," in the metric C, to the sets F_r, A_h, and Γ_ρ among all possible linear subspaces of dimension $2n$.* For the class B_k the following relation holds:

$$d_n(B_k) = \delta(B_k, P_n) = \frac{1}{n(n-1)\dots(n-k+1)}.$$

The value of the deviation of the class B_k from the space P_n of algebraic polynomials was computed in a recent work of K. I. Babenko ([6]).

In § 4 we give formulas for n-dimensional widths and their estimates for the classes $F_q^{E_s}$ of functions of s variables of smoothness q and for the class of functionals $F_\beta^{\widetilde{F}_1^{[a,b]}}$, for which there is no developed theory of approximation by any fixed subspace.

2. We proceed to the proof of the theorem. The n-dimensional space L_n appearing below is regarded as realized as a coordinate n-dimensional space with points $y = (y_1, \dots, y_n)$. The zero vector of L_n will be denoted by θ. In L_n Lebesgue measure is assumed to be specified.

Lemma. *Let U_N be a convex centrally symmetric body in the affine N-dimensional space R_N with center at zero; let U_{n+1} be the section of U_N by some fixed $(n+1)$-dimensional subspace $L^*_{n+1} \subset R_N$, $n+1 \le N$; let S_n be the boundary of U_{n+1}. Then for every n-dimensional subspace L_n there exists a point $x_0 = x_0(L_n) \in S_n$ such that the hyperplane $x_0 + L_n$ is a supporting hyperplane of U_n, i.e., for every $\varepsilon > 0$*

$$(1+\varepsilon)x_0 + L_n \cap U_N = 0. \tag{A}$$

The convex centrally symmetric body U_N specified in the lemma defines in R_N a Banach metric in which it itself is the sphere. Relation (A) means that in this metric $\rho(x_0, L_n) = 1$.

Thus our lemma can be reformulated as follows: *the n-dimensional width of an n-dimensional sphere does not decrease when it is embedded in a subspace of a larger number of dimensions.*

Proof of the lemma. The Banach metric defined by the body U_N determines a certain topology in R_N. Consider the following nonnegative function of two variables $x \in S_n$ and $y \in L_n$:

$$\varphi(x,y) = \begin{cases} \max\gamma \text{ over all } \gamma \text{ such that } x+y+\gamma\overline{U}_N \in \overline{U}_N, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } x+y \in \overline{U}_N, \\ 0, \qquad\qquad\qquad\qquad\qquad\quad\ \text{if } x+y \in R_N \setminus \overline{U}_N. \end{cases} \tag{1}$$

The function $\varphi(x,y)$ is continuous. Indeed, if $\varphi(x_0,y_0) = \alpha > 0$, then for any $\delta > 0$ and $x+y \in x_0+y_0+\delta U_N$ we obtain that $\alpha - \delta \le \varphi(x,y) \le \alpha+\delta$; whereas if $\varphi(x_0,y_0) = 0$, then for any δ and $x+y \in x_0+y_0+\delta U_N$ we obtain $0 \le \varphi(x,y) \le 2\delta$.

Now put

$$\psi(x) = \max_{y \in L_n} \varphi(x,y).$$

It is clear that $\psi(x)$ is continuous on S_n. The assertion of the lemma is equivalent to the existence of a point $x_0 \in S_n$ such that $\psi(x_0) = 0$. Suppose the contrary. Let $\psi(x) > 0$ for every point $x \in S_n$, and consequently, by the continuity of $\psi(x)$ and the compact-

* We note in this connection that the space T_{2n}, generally speaking, is not the only subspace satisfying the e

of S_n, $\psi(x) \geqslant \alpha > 0$. For each x consider the set $G(x) \subset L_n$, evidently nonempty, such that if $y \in G(x)$, then

$$\varphi(x, y) \geqslant \tfrac{1}{2}\alpha. \tag{2}$$

From (2) and the continuity of $\varphi(x, y)$ follows the closedness of the set $G(x)$. The set $G(x)$ is convex. Indeed, if $y_1, y_2 \in G(x)$, then this means that

$$x + y_1 + \tfrac{1}{2}\alpha\overline{U}_N \subset \overline{U}_N, \qquad x + y_2 + \tfrac{1}{2}\alpha\overline{U}_N \subset \overline{U}_N.$$

But then also

$$x + \gamma_1 y_1 + \gamma_2 y_2 + \tfrac{1}{2}\alpha\overline{U}_N \in \overline{U}_N, \qquad \gamma_1 + \gamma_2 = 1, \quad \gamma_1 \geqslant 0, \quad \gamma_2 \geqslant 0,$$

in view of the convexity of the set U_N. Moreover, the set $G(x)$ contains the zero vector θ for no x, since, by (1), $\varphi(x, \theta) = 0$, and, in consequence of the equality $\varphi(\bar{x}, -y) = \varphi(x, y)$, where $\bar{x}$ denotes the point symmetric to x with respect to the center (recall that U_N, and hence also S_n, are centrally symmetric), we obtain that the sets $G(x)$, as functions of x, are cosymmetric: $G(\bar{x}) = -G(x)$. Finally, from the continuity of the function $\varphi(x, y)$ it is easy to derive the continuous dependence of the set $G(x)$ on x. In view of the convexity of $G(x)$, all these sets are measurable.

Let $\eta(x)$ denote the center of gravity of the set $G(x)$. From what has been said above it follows that:

a) $\eta(x)$ is a vector of n dimensions, continuously depending on x;

b) $\eta(x) \neq \theta$ at no point $x \in S_n$;

c) $\eta(\bar{x}) = -\eta(x)$.

But, according to Borsuk' s theorem ([7]), on the n-dimensional sphere there cannot exist an n-dimensional vector field satisfying properties a)–c).*

We have arrived at a contradiction. The lemma is proved. The theorem follows simply from the lemma. Let L_n be some linear subspace. Consider the minimal subspace R_N such that it contains the subspace L^*_{n+1} of the theorem and the given subspace L_n. Let U_N denote the intersection of the unit sphere U with the subspace R_N.

According to the lemma, there will be found a point $x_0 \in \alpha\overline{U}_{n+1} \subset F_{n+1} \subset F$ such that, for any $\varepsilon > 0$,

$$(1+\varepsilon)x_0 + L_n \cap \alpha\overline{U}_N = \emptyset$$

and, consequently,

$$(1+\varepsilon)x_0 + L_n \cap \alpha U = 0. \tag{3}$$

It is easy to understand that relation (3) means that

$$\rho(x_0, L_n) \geqslant \frac{\alpha}{1+\varepsilon}.$$

In view of the arbitrariness of ε and L_n, we obtain what was required in the theorem.

3. We shall now derive relation (1). Let $f_0(x)$ denote the function from F_r for which

$$f_0^{(r)}(x) = \operatorname{sgn} \sin nx.$$

The function $f_0(x)$ has the following properties:

a)

$$\max_{x\in[0,2\pi]} |f_0(x)| = \frac{4}{\pi n^r} \sum_{m=0}^{\infty} \frac{(-1)^{m(r+1)}}{(2m+1)^{r+1}};$$

b) $f_0(x)$ assumes its maximum value with alternating signs at $2n$ consecutive points of the interval $[0, 2\pi)$:

$$x_k = \frac{k\pi}{n} + \frac{\pi\varepsilon}{2}$$

($k = 0, 1, \dots, 2n-1$), $\varepsilon = 1$, if r is even, and $\varepsilon = 0$, if r is odd. Denote by Δ_k the interval

$$k\pi \leqslant x \leqslant (k+1)\pi, \qquad k = 0, 1, \dots, 2n-1.$$

Consider $2n$ piecewise-constant functions:

$$\varphi_k(x) = 1, \quad \text{if } x \in \Delta_k; \qquad \varphi_k(x) = 0, \quad \text{if } x \notin \Delta_k.$$

Let L^*_{2n-1} denote the $(2n-1)$-dimensional space of functions $f(x)$ for which

$$f^{(r)}(x) = \sum_{k=0}^{2n-1} c_k \varphi_k(x), \qquad \sum_{k=0}^{2n-1} c_k = 0.$$

* We note that Borsuk' s theorem can be easily derived from the well-known theorem of Lusternik–Schnirelmann on the covering of spheres, as is done in ([8]), p. 107.

we prove that for every function $f(x) \in L^*_{2n-1}$ for which $\left|f^{(r)}(x)\right| \leqslant 1,\ 0 \leqslant x \leqslant 2\pi$, and on some Δ_k, $\left|f^{(r)}(x)\right| = 1$, the inequality

$$\max_{x\in[0,2\pi]} |f(x)| \geqslant \max_{x\in[0,2\pi]} |f_0(x)|. \tag{4}$$

is satisfied.

The relations (I) are derived automatically from (4) and the theorem. Suppose, without loss of generality, that $f^{(r)}(x) = +1$ for $x \in \Delta_0$. We split the function $f(x)$ into two: $f(x) = f_0(x) + f_1(x)$.

Assume that for some function $f(x)$ the inequality opposite to (4) holds. Then the function $f_1(x)$ will have $2n$ sign changes at the points x_k. Applying Lagrange' s theorem $r-1$ times, we obtain that the function $f^{(r-1)}(x)$ also has $2n$ sign changes, which is impossible, since it is piecewise linear on the $2n-1$ intervals $\Delta_k,\ k = 1, 2, \dots, 2n-1$, and constant on Δ_0.

The equalities (II) and (III) can be obtained on the basis of analogous arguments.

4. A real functional $F(f)$ on a set A from a Banach space R will be called satisfying the Hölder condition with exponent $\beta,\ 0 < \beta \leqslant 1$, if

$$|F(f) - F(f')| \leqslant \|f - f'\|^\beta.$$

By F_β^A we denote the set of all such functionals.

In particular, $F_\beta^{[a,b]}$ denotes the set of functions on the interval $[a, b]$ satisfying the usual Hölder condition

$$|f(x) - f(x')| \leqslant |x - x'|^\beta.$$

$F_\beta^{E_s}$ is the set of functionals satisfying the Hölder condition with exponent β on the s-dimensional cube $E_s :\ |x_k| \leqslant 1/2,\ k = 1, 2, \dots, s$, with metric

$$\|x - x'\| = \max_{1 \leqslant k \leqslant s} |x_k - x'_k|.$$

$\widetilde{F}_\beta^{[a,b]}$ is the set of functionals satisfying the Hölder condition with exponent β on the space $\widetilde{F}_1^{[a,b]}$ of functions $f(x)$ satisfying on $[a, b]$ the Lipschitz condition $(\beta = 1)$ with $f(a) = 0$.

Finally, let $F_q^{E_s}$ denote the set of functions $f(x_1, \dots, x_s)$ of s variables, of smoothness q in the sense of (9) (p. 32).

The following relations hold $(f(n) \asymp g(n)$ for positive functions f and g of n means that as $n \to \infty$, $f = O(g)$ and $g = O(f))$:

a) $d_n\left(F_\beta^{[a,b]}\right) = \left(\dfrac{b-a}{2n}\right)^\beta;$ b) $d_n\left(F_\beta^{E_s}\right) = \left(\dfrac{1}{2[n]^{1/s}}\right)^\beta;$

c) $d_n\left(F_q^{E_s}\right) \asymp \dfrac{1}{n^{q/s}};$ d) $d_n\left(\widetilde{F}_\beta^{[a,b]}\right) = \left\{\dfrac{b-a}{2\left([\log_2 n] + 1\right)}\right\}^\beta.$

Relation c) is a generalization of the relation $d_n(F_q^{E_1}) \asymp 1/n^q$, obtained by S. B. Stechkin (10).

In conclusion I express my deep gratitude to A. N. Kolmogorov for his attention to this work.

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Received
10 IX 1959

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Note: Figure translations are in progress. See original paper for figures.