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Abstract

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MATHEMATICS

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LAYERWISE EXTREMAL GROUPS

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The present paper arose in connection with a question posed to the author by S. N. Chernikov on the structure of groups satisfying the condition of Π-minimality for subgroups (Π is a given set of prime numbers).

A group $\mathfrak{G}$ **satisfies the condition of** Π**-minimality for subgroups*** if in it every chain

$$\mathfrak{G}_1 \supset \mathfrak{G}_2 \supset \cdots \supset \mathfrak{G}_k \supset \mathfrak{G}_{k+1} \supset \cdots$$

of subgroups terminates after a finite number of steps, having the following property: in each set $\mathfrak{G}_k \setminus \mathfrak{G}_{k+1}$ $(k = 1, 2, ...)$ there is an element G_k such that $G_k^{n_k} \in \mathfrak{G}_{k+1}$ for some natural number n_k, all prime divisors of which belong to the set Π (a chain with this property will be called Π-**decreasing**). From the definition of a group satisfying the condition of Π-minimality it follows, obviously, that it is periodic.

If a group $\mathfrak{G}$ satisfies the condition of Π-minimality for subgroups, then, without loss of generality, one may assume that Π is contained in the set of all prime divisors of the orders of elements of the group $\mathfrak{G}$, since numbers from the set Π not contained in this set impose no restrictions on the group $\mathfrak{G}$.

For periodic groups the definition of a Π-decreasing chain takes the following form: a chain of subgroups

$$\mathfrak{G}_1 \supset \mathfrak{G}_2 \supset \cdots \supset \mathfrak{G}_k \supset \mathfrak{G}_{k+1} \supset \cdots$$

of a periodic group $\mathfrak{G}$ is called Π-**decreasing** if in each set $\mathfrak{G}_k \setminus \mathfrak{G}_{k+1}$ $(k = 1, 2, ...)$ there is at least one Π-element.

If the set Π consists of only one prime number p, then we shall speak correspondingly of p-decreasing chains of subgroups and of groups satisfying the condition of p-minimality for subgroups (for brevity we shall say "groups satisfying the condition of p-minimality").

In the present paper we study groups satisfying the condition of p-minimality for every prime p (such groups we shall call groups satisfying the condition of

primary minimality). It turns out that, under certain sufficiently general restrictions, the class of such groups coincides with the class of layerwise extremal groups, i.e. groups in which every set of elements of one and the same order generates an extremal group (i.e. a finite group or a finite extension of the direct product of a finite number of quasicyclic groups ([1])).

1. We formulate several propositions on groups satisfying the condition of Π-minimality, which are used in obtaining the main result (Theorem 3). These propositions are also of independent interest.

Lemma 1. *A group* $\mathfrak{G}$ *satisfies the condition of* Π*-minimality if and only if this condition is satisfied by the subgroup generated by all Sylow* Π*-subgroups of the group* $\mathfrak{G}$*.*

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[*] The definition is due to S. N. Chernikov.

Theorem 1. Every locally finite group with the Π-minimality condition is an extension of a quasinilpotent group (i.e., a group having no proper subgroups of finite index) with the Π-minimality condition by a group all of whose Π-elements are contained in its finite normal divisor.

Corollary. A locally finite group having no quasinilpotent subgroups satisfies the Π-minimality condition if and only if the set of all its Π-elements is finite.

Theorem 2. A locally finite group $\mathfrak{G}$, every quasinilpotent subgroup of which is abelian, satisfies the Π-minimality condition if and only if the subgroup generated by all Sylow Π-subgroups of the group $\mathfrak{G}$ is extremal.

In the proof of this theorem the results of S. N. Chernikov' s paper ([2]) are used essentially.

Corollary. A locally finite group $\mathfrak{G}$ having an invariant system with finite factors satisfies the Π-minimality condition if and only if the subgroup generated by all Sylow Π-subgroups of the group $\mathfrak{G}$ is extremal.

2. From the results of S. N. Chernikov' s paper ([3]) it follows that the class of locally finite groups with the primary minimality condition is contained in the class of locally finite groups with special Sylow p-subgroups for all p. However, not every locally finite group with special (even finite) Sylow p-subgroups for all p satisfies the primary minimality condition.

Example. Let

$$\mathfrak{G} = (\{A_1\} \times \{A_2\} \times \cdots \times \{A_k\} \times \cdots) \cdot \{B\},$$

where

$$B^2 = A_k^{p_k} = 1, \qquad B^{-1} A_{kB} = A_k^{-1},$$

and $p_1, p_2, \dots, p_k, \dots$ are distinct odd primes.

Here all Sylow p-subgroups are finite; nevertheless the group $\mathfrak{G}$ does not satisfy the primary minimality condition, since it does not satisfy the 2-minimality condition.

Theorem 3. A locally finite group $\mathfrak{G}$ is a layerwise extremal group if and only if every countable subgroup of the group $\mathfrak{G}$ has a normal system with finite factors and satisfies the primary minimality condition.

Corollary. A locally finite H-group* (in particular, a locally soluble group) is a layerwise extremal group if and only if it satisfies the primary minimality condition.

3. Definition. A direct product of any periodic groups will be called **primarily thin** if, for every prime number p, only a finite number of direct factors contain p-elements.

Theorem 4. Layerwise extremal groups, and only they, are subgroups of primarily thin direct products of extremal groups.

Remark 1. Theorem 4 is analogous to S. N. Chernikov' s theorem ([4]) on the embeddability of a thin layerwise finite group in a primarily thin direct product of finite groups.

Remark 2. A layerwise extremal group does not necessarily decompose into a direct product of extremal groups, since there exist thin layerwise finite groups that do not decompose into a direct product of finite groups ([5]).

Corollary. Layerwise finite groups, and only they, are subgroups of primarily thin direct products of extremal groups that are central extensions of their maximal complete subgroups.

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* The definition of an H-group is given in paper ([1]).

From Theorem 4 one can obtain a number of properties of layer-extremal groups analogous to the corresponding properties of layer-finite groups.

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[5] S. N. Chernikov, Mat. sborn., **22**, 101 (1948).

Note: Figure translations are in progress. See original paper for figures.