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Abstract

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ON THE BASIC CONCEPTS OF ALGEBRAIC TOPOLOGY

AN AXIOMATIC DEFINITION OF COHOMOLOGY GROUPS

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In this note the basic concepts of the theory of cohomology groups are axiomatized. The essential difference between the axiomatization proposed by us and the classical Eilenberg-Steenrod axiomatization ([1]) consists in the fact that we abandon the consideration of relative groups. In our opinion, the use of relative groups in substantive geometric questions is superfluous and should be excluded. Such an exclusion considerably simplifies the exposition of the theory and in most cases makes the proofs simpler and clearer. At the same time, the exclusion of relative groups from consideration makes the excision axiom unnecessary; owing to this, as we intend to show in the next note, a complete parallelism (duality) can be achieved in the axiomatization of cohomology groups and homotopy groups. An essential feature of the axiomatization given here is also that cohomology groups are defined for spaces with distinguished points, so that the zero-dimensional cohomology groups turn out to be reduced. The expediency of such an approach to cohomology groups is substantiated in a series of notes by B. Eckmann and P. Hilton ([2]).

Thus, our constructions will be carried out in the category $\mathfrak{A}_0$, whose objects are topological spaces with distinguished points, and whose mappings are continuous mappings carrying distinguished points into distinguished points. The distinguished point of a space X will be denoted by the symbol o_X, or simply o. By the same symbol we shall denote the space consisting of a single point.

For any space X we shall denote by the symbol IX the topological product $[0,1] \times X$, in which the segment $[0,1] \times o_X$ has been contracted to the point $o = o_{IX}$. The mapping $X \to IX$, defined by the correspondence $x \to (t, x)$, will be denoted by q_t. For any mapping $f : X \to Y$, by If we shall denote the mapping $IX \to IY$ defined by the correspondence $q_t(x) \to q_t(f(x))$.

A sequence of mappings

$$\cdots \to X_{i-1} \xrightarrow{f_{i-1}} X_i \xrightarrow{f_i} X_{i+1} \to \cdots$$

will be called **exact** if, for every i, the image $\operatorname{Im} f_{i-1} = f_{i-1}(X_{i-1})$ of the mapping f_{i-1} coincides with the kernel $\operatorname{Ker} f_i = f_i^{-1}(o)$ of the mapping f_i.

An exact sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z \to o$$

will be called a **cofiltration** (with respect to a certain subcategory Ω of the category $\mathfrak{A}_0$) if, for every space Q of the category Ω and any mappings $\Lambda : IX \to Q$, $\mu : Y \to Q$ (of the category $\mathfrak{A}_0$) connected by the relation $\Lambda \circ q_0 = \mu \circ f$, there exists a mapping $M : IY \to Q$ such that $\mu = M \circ q_0$ and $\Lambda = M \circ If$. The space X is called the **cobase**, the space Y is called the **space of the cofiltration**, and the space Z the **cofiber**.

We shall say that the mappings $\lambda : X \to X'$, $\mu : Y \to Y'$, $\nu : Z \to Z'$ constitute a mapping of the cofibration

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

into the cofibration

$$X' \xrightarrow{f'} Y' \xrightarrow{g'} Z',$$

if $\mu \circ f' = f \circ \lambda$ and $\nu \circ g' = g \circ \mu$.

For definiteness, in what follows one may assume that $\mathfrak{L} = \mathfrak{A}_0$.

A subcategory $\mathfrak{A}$ of the category $\mathfrak{A}_0$ will be called admissible (for the theory of cohomology groups) if:

1) the category $\mathfrak{A}$ contains the zero-dimensional sphere S^0;
2) together with each space X, the category $\mathfrak{A}$ contains the space IX and the mappings $q_0, q_1 : X \to IX$; together with each mapping f, the category $\mathfrak{A}$ contains the mapping If;
3) if in the cofibration

$$X \xrightarrow{f} Y \xrightarrow{g} Z \qquad (*)$$

the spaces X and Y belong to the category $\mathfrak{A}$, then the space Z also belongs to the category $\mathfrak{A}$; if in the cofibration $(*)$ the spaces X and Z belong to the category $\mathfrak{A}$, then the space Y also belongs to the category $\mathfrak{A}$; if in the cofibration $(*)$ all three spaces X, Y, and Z belong to the category $\mathfrak{A}$, then the mappings f, g also belong to the category $\mathfrak{A}$.

It is easy to see that any admissible category contains all finite polyhedra (i.e., spaces admitting finite cellular decompositions).

In what follows we shall assume that some admissible category $\mathfrak{A}$ has been chosen and fixed. The spaces and mappings of this category will be called admissible. Cofibrations $(*)$ with admissible spaces X, Y, Z will also be called admissible.

Admissible mappings $f_0, f_1 : X \to Y$ are called homotopic (in $\mathfrak{A}$) if there exists an admissible mapping $F : IX \to Y$ such that

$$F \circ q_i = f_i, \quad i = 0, 1.$$

Suppose that for every integer n a contravariant functor H^n is given, defined in the category $\mathfrak{A}$ and taking values in the category of abelian groups and their homomorphisms*. Then on the category of all admissible cofibrations (and their admissible mappings) three functors $H^n_I, H^n_{\mathrm{II}}, H^n_{\mathrm{III}}$ are defined, the first of which assigns to the cofibration $(*)$ the group $H^n(X)$, the second—the group $H^n(Y)$, and the third—the group $H^n(Z)$. (However, we shall not need the functor H^n_{II}.)

We shall say that the functors H^n constitute a theory of cohomology groups if, for every n, a natural transformation δ^n of the functor H^n_I into the functor H^{n+1}_{III} is given and the following three axioms are satisfied:

1^H. The groups $H^n(S^0)$ for $n \neq 0$ are trivial.

2^H. For any admissible mappings $f, g : X \to Y$ homotopic to each other, the homomorphisms

$$H^n(f),\ H^n(g) : H^n(Y) \to H^n(X)$$

coincide.

3^H. For every admissible cofibration

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

there is an exact sequence

$$\cdots \to H^n(Z) \xrightarrow{H^n(g)} H^n(Y) \xrightarrow{H^n(f)} H^n(X) \xrightarrow{\delta^n} H^{n+1}(Z) \to \cdots.$$

The group $H^0(S^0)$ is called the coefficient group of the cohomology-group theory under consideration.

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* Instead of abelian groups one may also consider arbitrary modules or commutative topological groups; see ([1]), Chap. I, Sec. 2.

It is easily verified that these axioms are satisfied, for example, by the theory of singular cohomology groups with an arbitrary coefficient group (the **existence theorem**). It can also be shown that, for finite polyhedra, axioms 1^H—3^H uniquely determine the cohomology groups (the **uniqueness theorem**).

The same axioms (with obvious modifications) also describe the **homology groups**. However, we believe (and shall explain this in subsequent publications) that homology theory does not have so primary a character as cohomology theory.

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CITED LITERATURE

[1] N. Steenrod, S. Eilenberg, *Foundations of Algebraic Topology*, Moscow, 1958.
[2] B. Eckmann, P. Hilton, Homotopy groups and duality, *Translations. Mathematics*, 4, No. 3, 3 (1960).

Note: Figure translations are in progress. See original paper for figures.