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Abstract

Full Text

ASTRONOMY

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EXISTENCE OF OSCILLATING MOTIONS IN THE THREE-BODY PROBLEM

(Presented by Academician A. N. Kolmogorov, 4 III 1960)

In this paper a system of three bodies moving under the action of the forces of mutual Newtonian attraction is constructed, in which the phenomenon of oscillation occurs, i.e., as $t \rightarrow \infty$ the distance between bodies (1) and (2) is bounded, while the distances between each of them and body (3) are unbounded and do not tend to infinity. This question was posed by the theorems of Lagrange and Poisson on the invariance of the major axes in the corresponding approximations^(1,2). First we shall construct such a system in the restricted three-body problem, when body (3) has infinitely small mass, and then pass to a finite, arbitrarily small mass of body (3).

Let there be in space an inertial rectangular coordinate system $OXYZ$. Consider a system of two bodies (1) and (2) of equal arbitrary mass M , whose center of gravity is always at the origin O , and which move under the action of the forces of mutual attraction along ellipses lying in the plane OXY , the major axes and eccentricities of which are equal to arbitrary R and $e \neq 0$. The state of this system is determined by the angle φ formed by the radius vector r of body (1) with the axis X . We shall now define a system of three bodies (1), (2), and (3), depending on two parameters $\varphi(0)$ and $v(0)$, as follows: at $t = 0$ we prescribe the angle $\varphi(0)$ and the coordinates of body (3) of infinitely small mass equal to $0, 0, 0$, and the components of its velocity equal to $0, 0, v(0)$.* We shall prove that for every value of $\varphi(0)$ and any sequence of numbers $\{S_k\}$ (in particular, for any sequence tending to infinity) there exists such a value $v(0)$ that in the system $\varphi(0), v(0)$, for $t > 0$, body (3) passes through the point O an infinite number of times, receding from it after the k -th passage to a distance greater than S_k .

It is easy to see that body (3) for all t lies on the axis Z . Further, for any $\varphi(0)$ there exists a minimal $v_\varphi(0)$ for which the coordinate $z(t)$ of body (3) for $t > 0$ increases monotonically to plus infinity.

Lemma 1. For any $\varphi(0)$ and $t > 0$ there exists such a positive $v(0) < v_\varphi(0)$ that body (3) returns for the first time to the point O at the moment of time t .

First we shall prove that if $v(0) < v_\varphi(0)$ is sufficiently close to $v_\varphi(0)$, then the time of the first return will be arbitrarily large. Indeed, if $v(0) < v_\varphi(0)$ is

sufficiently close to $v_\varphi(0)$, then, on the basis of the continuous dependence of the solution of the equation of motion of body (3) on the initial conditions, the maximal distance σ of body (3) from the point O will be arbitrarily large. In the motion “upward,” at all points of the segment $0, \sigma$, the magnitude of the velocity of body (3) is less than $v(0)$, and therefore the time of the first return tends to infinity together with σ .

* Systems with this kind of symmetry with respect to the axis Z were considered by A. N. Kolmogorov.

Let us now take such a $v_1(0)$ for which the time of the first return $t' > t$. When the initial velocity of body (3) is varied from 0 to $v_1(0)$, the time of its first return to the point O will vary continuously from 0 to t' , and the assertion of the lemma follows from the fact that a continuous function assumes all its intermediate values.

Lemma 2. There exists a number S (depending on R and $\varepsilon \neq 0$) such that: if in the system $\varphi(0), v(0)$ at time $t_1 < 0$ bodies (1) and (2) are at the minimum distance, and at time $t_2 > 0$ at the maximum distance; if t_2 is the nearest to t_1 among the greater instants having this property; if the segment $z(t_1), z(t_2)$ is bisected by the point O , and if, as time decreases from 0, the coordinate $z(t)$ increases monotonically to a value greater than S , then, as time increases from 0, the coordinate $z(t)$ decreases monotonically to minus infinity.

First we shall prove that if the conditions of the lemma are fulfilled with sufficiently large S , then $z(t_1) = |z(t_2)| > R\sqrt[3]{2}$. Indeed, the acceleration experienced by body (3) at the point z at time t , equal to

$$\frac{2Mz}{[z^2 + r^2(t)]^{3/2}},$$

is greater in absolute value than $\frac{2M}{z^2\sqrt{6}}$ for

$$z > R\sqrt[3]{2} > r(t)\sqrt[3]{2}.$$

Therefore the magnitude of the velocity of body (3) at the point $z = R\sqrt[3]{2}$, for sufficiently large S , is greater than $\sqrt{4M/3R}$. Since at all points of the segment

$$-R\sqrt[3]{2}, R\sqrt[3]{2}$$

the velocity of body (3) is greater than at its end $R\sqrt[3]{2}$, the time it takes to traverse it is less than

$$R\sqrt{\frac{R}{M}} \frac{3}{\sqrt{2}}.$$

This time is less than the half-period $T/2 = t_2 - t_1$ of the motion of bodies (1) and (2) along the ellipses, which is, as is known,

$$\frac{\pi R}{\sqrt{2}} \sqrt{\frac{R}{M}},$$

and therefore

$$z(t_1) = |z(t_2)| > R\sqrt[3]{2}.$$

At every point z of the segment $0, z(t_1)$, the acceleration of body (3) will be greater in absolute value than its acceleration at the point $-z$, since in the first case bodies (1) and (2) are at a smaller distance than in the second. Hence it follows that $|v(t_2)| - |v(t_1)| > \alpha$. For each $t_1 - t$, $t > 0$, we shall compare the acceleration $a(t)$ of body (3), located at the point $z(t_1 - t)$, with the acceleration $a^*(t)$ produced by bodies (1) and (2) at time $t_2 + t$ at the point $-z(t_1 - t)$, symmetric with respect to the plane OXY . We shall prove that

$$W(t) = \int_0^t |a(t')| dt' > \int_0^t a^*(t') dt' = W^*(t), \quad \int_0^t |W(t')| dt' > \int_0^t W^*(t') dt' \quad (1)$$

for t up to which $z(t_1 - t)$ increases monotonically, and the first inequality moreover for $t = nT$, where n is an integer.

The difference $\varphi(t) = |a(t)| - a^*(t)$, equal to

$$\frac{2Mz(t_1 - t)}{[z^2(t_1 - t) + r^2(t_1 - t)]^{3/2}} - \frac{2Mz(t_1 - t)}{[z^2(t_1 - t) + r^2(t_2 + t)]^{3/2}},$$

will be positive for $0 < t < T/4$, since for these t

$$r(t_1 - t) < r(t_2 + t),$$

negative for $T/4 < t < T/2$, negative for $T/2 < t < 3T/4$, and positive for $3T/4 < t < T$. For each τ_1 , $0 < \tau_1 < T/4$, the corresponding pair of radius vectors $r_1(t_1 - \tau_1)$ and $r_2(t_2 + \tau_1)$ is encountered once in each of the other parts at the instants of time

$$\tau_2 = T/2 - \tau_1, \quad \tau_3 = T/2 + \tau_1, \quad \tau_4 = T - \tau_1,$$

and here

$$\tau_2 - \tau_1 = \tau_4 - \tau_3.$$

The coordinates of body (3) at these instants are z_1, z_2, z_3, z_4 .

Consider the values $|\varphi(\tau_k)| = f(z_k)$, $k = 1, 2, 3, 4$, where

$$f(z) = \frac{2Mz}{[z^2 + r_1^2]^{3/2}} - \frac{2Mz}{[z^2 + r_2^2]^{3/2}}.$$

Since $R\sqrt{3}/2 < z_1 < z_2 < z_3 < z_4$, $z_2 - z_1 > z_4 - z_3$, and for $z > R\sqrt{3}/2$, $f'(z) < 0$, $f''(z) > 0$, it follows that $\varphi(\tau_1) > |\varphi(\tau_2)| > |\varphi(\tau_3)| > \varphi(\tau_4)$ and $\varphi(\tau_1) - |\varphi(\tau_2)| > |\varphi(\tau_3)| - \varphi(\tau_4)$.

From these properties of $\varphi(t)$, which also hold for the following periods, the inequalities (1) follow. From these inequalities it follows that $z(t_1 - t) < |z(t_2 + t)|$

for those t for which $z(t_1 - t)$ increases monotonically. Suppose the contrary, and let τ be the first value $t > 0$ for which $z(t_1 - \tau) = |z(t_2 + \tau)|$. For every $t < \tau$ the acceleration of body (3) at $t_2 + t$ is less than $a^*(t)$, since $z(t_2 + t) < -z(t_1 - t)$ and the function

$$\frac{2Mz}{[z^2 + r^2(t_2 + t)]^{3/2}}$$

decreases for $z > R\sqrt{3}/2 > r(t)\sqrt{3}/2$. But even if body (3) had a greater acceleration, equal to $a^*(t)$, then, by virtue of the inequalities (1) and because $|v(t_2)| - |v(t_1)| > \alpha$, the equality $z(t_1 - \tau) = |z(t_2 + \tau)|$ could not occur.

From the inequalities (1) it follows that, as time increases, body (3) at all sufficiently distant points will have velocity less than $-\alpha$, and therefore, if S is also greater than $4/\alpha^2$, then $z(t)$ decreases monotonically to minus infinity. Lemma 2 is proved.

Existence of oscillation. Thus, for the given $\varphi(0)$ and $\{S_k\}$ it is required to find $v(0)$. Take $v_\varphi(0) > 0$ and an interval O_1 , whose right endpoint is the value v_φ , so small that for every $v(0)$ belonging to O_1 , body (3) moves monotonically away from the point O to a distance greater than S (see Lemma 2).

Take the moments of time t_n and $t_n + T/2$, at which bodies (1) and (2) are at the minimal and maximal distances, sufficiently large so that the values $v_n(0)$ and $v'_n(0)$ found for them by Lemma 1 belong to O_1 . In the system $\varphi(0), v_n(0)$, the segment traversed by body (3) during the time from t_n to $t_n + T/2$ lies below the point O , while in the system $\varphi(0), v'_n(0)$ it lies above; and since its position and length vary continuously as $v(0)$ varies, there will be a $v''(0)$, lying between $v_n(0)$ and $v'_n(0)$, such that in the system $\varphi(0), v''(0)$ this segment will be bisected by the point O . Consequently, by Lemma 2, in the system $\varphi(0), v''(0)$, body (3), after its first return to the point O , moves monotonically to minus infinity. In the system $\varphi(0), v_n(0)$, the positions of body (3) at the moments of time $t_n + t$ and $t_n - t$ will be symmetric with respect to the plane OXY , and therefore, after its first return to the point O , body (3) will “descend” only to a finite height, equal to the height of the first “ascent.”

Varying $v(0)$ from $v_n(0)$ to $v''(0)$, take the first value $v_2(0)$ for which body (3), after its first return to the point O , moves monotonically to minus infinity, and take so small an interval O_2 , whose endpoint is $v_2(0)$, that for every $v_2(0)$ belonging to O_2 , body (3), after its first return to the point O , moves away from it only to a finite distance, but greater than S and S_1 from $\{S_k\}$.

In an analogous way, in O_2 we find such a $v_3(0)$ that, for the value $v(0)$ equal to $v_3(0)$, body (3), after its second return to the point O , moves monotonically to plus infinity, and such an interval O_3 , whose endpoint is $v_3(0)$, that for values $v(0)$ belonging to O_3 , body (3), after its second return to the point O , moves away from it to a finite distance greater than S_2 and S . Here $\overline{O_3} \subset O_2$, i.e. O_3 , together with its endpoints, is contained in O_2 .

Continuing this process further, we obtain a decreasing sequence of intervals $O_k, \bar{O}_{k+1} \subset O_k$, where O_{k+1} has the property that, for values $v(0)$ belonging to O_{k+1} , body (3) returns k times...

returns to the point O , after which it recedes from it only to a finite distance, but greater than S_k , while for the value $v(0)$ equal to one of the endpoints O_k , body (3), after the k -th return, goes off to infinity. Hence it follows that there exists a $v(0)$, belonging to all O_k , which is the desired one, i.e., in the system $v(0), \varphi(0)$ body (3) will perform oscillations if $S_k \rightarrow \infty$.

The case of finite mass of body (3). Let us prescribe, for $t = 0$, the positions and velocities of bodies (1), (2), (3) in the inertial coordinate system $OXYZ$ to be the same as in the system $\varphi(0), v(0)$; the masses of bodies (1) and (2) are, as before, equal to M , and the mass of body (3) is equal to m . We obtain a system depending on three parameters $\varphi(0), v(0)$, and m . We shall prove that there exists an α such that, for any $\varphi(0)$, $m < \alpha$, and $\{S_k\}$, there exists a $v(0)$ such that, in the system $\varphi(0), v(0), m$, for $t > 0$, body (3) passes infinitely many times through the center of gravity O' of bodies (1) and (2), receding from it after the k -th passage to a distance greater than S_k , while the distance between bodies (1) and (2) is always less than $2R$.

Lemma 3. *For any $\delta > 0$ and $T' > 0$, there exists an $\alpha' > 0$ such that in any system $\varphi(0), v(0), m$ with $m < \alpha'$, for any value of time τ_0 , the motion of bodies (1) and (2) for $\tau_0 < t < \tau_0 + T'$ differs by less than δ from the motion of bodies (1) and (2) along certain ellipses with parameters R and e (equal to the parameters of the "principal" ellipses) under the action only of their mutual attraction (without the influence of body (3)).*

We divide all times $t > 0$ into intervals of arbitrary length T^* , and on each of them approximate the motions of bodies (1) and (2) in the system $\varphi(0), v(0), m$ by elliptic motions. We choose the instants of time t_n at which the distances of bodies (1) and (2) are minimal and maximal in these approximations. Then, if m is sufficiently small, for any $\varphi(0)$ and $v(0)$, by virtue of Lemma 3 we have $t_n = t_0 + nT/2 + \theta\varepsilon$, where n ranges over all integers, $|\theta| < 1$, and ε is arbitrarily small.

The scheme for constructing an example of oscillation remains the same as in the restricted problem. In the system $\varphi(0), 0, m$ we choose sufficiently large t_n and $t'_n = t_n + T/2 + \varepsilon$. As $v(0)$ increases from 0, the return time of body (3) to the point O' will grow to infinity, while the times t_n and t'_n of these selected minimum and maximum instants change discontinuously within bounded limits, but the magnitudes of their jumps are arbitrarily small if m is sufficiently small. Therefore there exist such $v_n(0)$ and $v'_n(0)$ for which the return times will differ arbitrarily little from the corresponding t_n and t'_n . The construction continues further, and Lemma 2 for the system $\varphi(0), v(0), m$, under the assumption that m is sufficiently small, remains valid.

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