Reports of the Academy of Sciences of the USSR

Corresponding Member of the USSR Academy of Sciences S. N. Mergelyan

1960

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR

1960. Volume 132, No. 2

MATHEMATICS

Corresponding Member of the USSR Academy of Sciences S. N. Mergelyan

ON BEST APPROXIMATIONS WITH A WEIGHT ON THE LINE

Let $0 < h(x) < 1$, and let $f(x)$ be continuous on the axis $-\infty < x < \infty$, and

$$\lim_{|x|\to\infty} h(x)f(x) = 0. \tag{1}$$

By $E_n(h, f)$ we denote the lower bound of the numbers

$$\sup_{-\infty<x<\infty} h(x)|f(x) - P_n(x)|$$

in the class of all polynomials of degree not exceeding n.

In the case when the relation $\lim_{n\to\infty} E_n(g, f) = 0$ holds for every continuous function $f(x)$ satisfying condition (1), one says that the system of polynomials is complete with weight $h(x)$ on the line $-\infty < x < \infty$. The rate of convergence to zero of the numbers $E_n(h, f)$ as $n \to \infty$ in the case of completeness depends on the properties of the weight function $h(x)$ and of the function $f(x)$ being approximated.

The purpose of the present note is to estimate the rate of decrease to zero of $E_n(h, f)$, i.e., to establish a theorem on the best weighted approximation for a certain class of weight functions $h(x)$.

With respect to $h(x)$ we shall assume, first, that $h(x)$ admits the representation

$$h(x) = h(0) \exp\left\{-\int_0^{|x|} \frac{\omega(t)}{t}\, dt\right\}, \qquad -\infty < x < \infty, \tag{2}$$

in which $0 \le \omega(t) < \infty$, $\omega(t)$ increases monotonically, and, secondly, that the integral

$$\int_{-\infty}^{\infty} \frac{\ln h(\xi)}{1+\xi^2}\, d\xi$$

diverges.

The first requirement concerns the regularity of the decrease of $h(x)$ at infinity; the second requirement is natural and does not diminish the generality of the considerations, since it is a necessary and sufficient condition for completeness for functions $h(x)$ satisfying the first requirement.

As the function to be approximated, in the present note we consider the Cauchy kernel, i.e., a function of the form $\frac{1}{x-a}$, where $\operatorname{Im} a \neq 0$.

Introduce the notation: $H(x) = \frac{1}{h(x)}$; $p(x) = \ln H(x)$; $q(x)$ is the function inverse to the function $p(x)$; θ is an arbitrary number in the range $0 < \theta < 1$, and $\varkappa = \frac{\theta^{2n+2}}{1-\theta^2}$. Let $\delta = \ln \frac{19h(0)}{eh(1)}$.

Theorem. There exists an absolute constant $C > 0$ such that the inequality

$$E_n\left(h, \frac{1}{x-a}\right) < C \frac{h^2(1)}{h(0)} \frac{1-\varkappa}{|\operatorname{Im} a|} \exp\left\{ \frac{|\operatorname{Im} a|}{\pi(1+|a|^2)} \int_0^{e^{-10} q(h-\delta)} \frac{P(\xi)}{1+\xi^2}\, d\xi \right\}$$

is valid for all n satisfying the conditions $\varkappa < 1$ and $n > \delta$.

Proof. We may evidently assume that $\alpha = \operatorname{Im} a > 0$. Let $M_{n,h}(z)$ denote $\sup |P(z)|$ over all polynomials of degree not exceeding n that satisfy on the real axis the inequality

$$h(x)|P(x)| \leqslant 1 + |x|. \tag{3}$$

We shall now estimate $M_{n,h}(z)$ from below, from which the theorem will easily follow. First suppose that the function $H(x)$ is an entire even function with nonnegative coefficients

$$H(x) = \sum_{k=0}^{\infty} a_k x^{2k}, \qquad a_0 \geqslant 1, \qquad a_k \geqslant 0.$$

Put

$$\mu_{2k} = \max_{x>0} [H(x)]^{-1/2k} x.$$

From Cauchy’ s inequality it follows that

$$0 \leqslant a_k \leqslant \min_{x>0} \frac{H(x)}{x^{2k}} = \left(\frac{1}{\mu_{2k}}\right)^{2k}, \qquad k \geqslant 0.$$

Therefore, for the remainders of the Taylor series we shall have

$$0 \leqslant H(x) - P_{2n}(x) = \sum_{k=n+1}^{\infty} a_k x^{2k} \leqslant \sum_{k=n+1}^{\infty} \left(\frac{x}{\mu_{2k}}\right)^{2k}.$$

If $\lambda_n = \theta\mu_{2n+2}$, then, by virtue of the monotonicity of the numbers μ_{2n}, everywhere on the interval $|x| \leqslant \lambda_n$

$$H(x) - P_{2n}(x) \leqslant \sum_{k=n+1}^{\infty} \theta^{2k} = \chi.$$

The polynomial $P_{2n}(z)$ has $2n$ zeros, symmetrically situated with respect to the real axis, outside this axis.

Let $Q_n(z)$ denote the polynomial of degree n whose zeros coincide with the zeros of $P_{2n}(z)$ situated in the lower half-plane. By a suitable normalization of $Q_n(z)$ one can achieve the identity

$$Q_n(z)\overline{Q_n}(z) \equiv P_{2n}(z),$$

in which $\overline{Q_n}(z)$ has coefficients complex conjugate to the coefficients of the polynomial $Q_n(z)$. On the interval $-\lambda_n \leqslant x \leqslant \lambda_n$, from the inequality

$$H(x) - P_{2n}(x) \leqslant \chi$$

and taking into account that $H(x) \geqslant 1$, we obtain

$$|Q_n^2(x)| = |Q_n(x)\overline{Q_n}(x)| \geqslant (1-\chi)H(x).$$

Moreover, everywhere on the real axis

$$|Q_n^2(x)| = |P_{2n}(x)| \leqslant 1.$$

Since all zeros of $Q_n(z)$ are situated in the region $\operatorname{Im} z < 0$, the Poisson formula for representing harmonic functions in the upper half-plane is applicable to the function $\ln|Q_n^2(z)|$. From this formula, for $z = a$, it follows that

$$\ln|Q_n^2(a)| = \frac{\alpha}{\pi}\int_{-\infty}^{\infty}\frac{\ln|Q_n^2(\xi)|}{(\xi-\gamma)^2+\alpha^2}\,d\xi.$$

Taking into account the two inequalities for $Q_n^2(x)$ given above, as well as the easily proved inequality

$$\frac{1}{1+|a|^2}\,\frac{1}{1+\xi^2} \leqslant \frac{1}{(\xi-\gamma)^2+\alpha^2},$$

we find

$$\ln|Q_n^2(a)| \geq \frac{a}{\pi}\int_{-\lambda_n}^{\lambda_n}\frac{\ln(1-x)H(\xi)}{(\xi-\gamma)^2+a^2}\,d\xi \geq \frac{a}{\pi(1+|a|^2)}\int_0^{\lambda_n}\frac{\ln H(\xi)}{1+\xi^2}\,d\xi + \ln(1-x).$$

From the definition of μ_{2n} it follows that, for any $x > 0$,

$$\mu_{2n} \geq x/\sqrt[2n]{H(x)}.$$

Choosing x from the condition $\ln H(x) = 2n$, we obtain, in particular,

$$\mu_{2n} \geq e^{-1}q(2n),$$

whence

$$\lambda_n \geq \theta e^{-1}q(2n+2).$$

The polynomial $Q_n^2(z)$ of degree $2n$ satisfies condition (3), and therefore

$$M_{2n,h}(a) \geq |Q_n^2(a)|.$$

But, obviously,

$$M_{2n+1,h}(a) \geq M_{2n,h}(a),$$

hence, irrespective of the parity of the number $n \geq 0$, we have

$$M_{n,h}(a) \geq (1-x)\exp\left\{\frac{a}{\pi(1+|a|^2)}\int_0^{\theta e^{-1}q(n+1)}\frac{\ln H(\xi)}{1+\xi^2}\,d\xi\right\}. \qquad (4)$$

We now pass to the derivation of an analogous inequality without the assumption that $H(x)$ is an entire, even function with nonnegative coefficients, while retaining, however, the requirement that $h(x)$ be representable in the form (2).

It was established by V. S. Videnskii (1) that, for any function $h(x)$ of the form (2), there exists a function $F(x)$

$$F(x) = \sum_{k=0}^{\infty} b_k x^{2k}, \quad b_0 > 0, \quad b_k \geq 0, \tag{5}$$

satisfying, for sufficiently large $|x|$, the inequalities

$$x^{-2} F\left(\frac{x}{2}\right) < H(x) < F(x).$$

Subsequently Videnskii improved this result, proving under the same assumptions the existence of a function $F(x)$ of the form (5), with the condition $F(0) \geq 1$, for which the inequalities

$$\frac{1}{19} F(x) < H(x) < x^2 F(x) \quad \text{for } |x| > 1 \tag{6}$$

hold. We shall use the existence of this function $F(x)$; for it, on the whole axis, the relations

$$\frac{1}{19} \frac{H(0)}{H(1)} F(x) < H(x) < x^2 F(x) + H(1) \tag{7}$$

plainly hold.

It follows from the left inequality that any polynomial $R(x)$ satisfying

$$|R(x)| \leq F(x)(1 + |x|)$$

satisfies the inequality

$$\left| \frac{H(0)}{19 H(1)} R(x) \right| \leq H(x)(1 + |x|), \quad -\infty < x < \infty,$$

and therefore

$$M_{n,h}(\alpha) \geq \frac{h(1)}{19 h(0)} M_{n,1/F}(a).$$

But for the function $1/F(x)$ the same conditions hold under which inequality (4) was derived. We may therefore, in estimating $M_{n,1/F}(a)$, use inequality (4), with the obvious replacement in the right-hand side of the function $\ln H(x)$

by $\ln F(x)$ and $q(x)$ by $\varphi(x)$, where $\varphi(x)$ is the function inverse to $\ln F(x)$. Thus, we have a lower estimate for $M_{n,h}(a)$ in terms of the function $F(x)$:

$$M_{n,h}(a) \geqslant \frac{h(1)}{19h(0)}(1-x)\exp\left\{\frac{a}{\pi(1+|a|^2)}\int_0^{\theta e^{-1}\varphi(n+1)}\frac{\ln F(\xi)}{1+\xi^2}\,d\xi\right\}. \quad (8)$$

It remains, with the help of inequalities (6) and (7), to pass to an estimate in terms of $h(x)$. From the left-hand side of (7) we find the estimate

$$\varphi(x+1) \geqslant q(x-\delta), \quad \text{where } \delta = \ln\frac{19h(0)}{eh(1)}.$$

Writing inequality (8) briefly in the form

$$M_{n,h}(a) \geqslant A\exp\left\{B\int_0^L \frac{\ln F(\xi)}{1+\xi^2}\,d\xi\right\},$$

we have

$$M_{n,h}(a) \geqslant A\exp\left\{B\int_0^L \frac{\ln H(\xi)}{1+\xi^2}\,d\xi\right\}\cdot\exp\left\{B\int_0^L \frac{\ln F(\xi)-\ln H(\xi)}{1+\xi^2}\,d\xi\right\},$$

where $B \leqslant 1/\pi$ and

$$\exp\left\{B\int_0^2 \frac{\ln F(\xi)-\ln H(\xi)}{1+\xi^2}\,d\xi\right\} \geqslant$$

$$\geqslant \exp\left\{-B\ln H(1)\int_0^1\frac{d\xi}{1+\xi^2} - 2B\int_1^\infty \frac{\ln\xi}{1+\xi^2}\,d\xi\right\} \geqslant \theta_1[h(1)]^{\theta_2};$$

here $\theta_1 > 0$, $\theta_2 > 0$ are absolute constants, $\theta_2 < 1$. Taking these relations into account, we finally obtain

$$M_{n,h}(a) \geqslant C_0\frac{h^2(1)}{h(0)}(1-x)\exp\left\{\frac{a}{\pi(1+|a|^2)}\int_0^{\theta e^{-1}q(n-\delta)}\frac{P(\xi)}{1+\xi^2}\,d\xi\right\}, \quad n \geqslant \delta,$$

where $C_0 > 0$ is an absolute constant.

Passing to best approximations, let us note that if $P(z)$ is an arbitrary polynomial of degree $n+1$ satisfying condition (3), then for the polynomial

$$S_n(z) = \frac{P(z) - P(a)}{(a-z)P(a)}$$

we shall have

$$\sup h(x) \left| S_n(x) - \frac{1}{x-a} \right| \leqslant \frac{C_1}{a} \frac{1}{|P(a)|},$$

where $C_1 > 0$ is an absolute constant.

The right-hand side can be made arbitrarily close to

$$\frac{C_1}{a} \frac{1}{M_{n+1,h}(a)},$$

while the left-hand side is then always not less than $E_n\left(h, \frac{1}{x-a}\right)$. The theorem is proved.

From this theorem one can obtain estimates of the best weighted approximation for various classes of approximated functions $f(x)$, which can be characterized by one or another differential property. For this it is necessary first to approximate the function $f(x)$ by linear combinations of the functions $\frac{1}{x-a_1}, \dots, \frac{1}{x-a_m}$ and to choose in an optimal way the number of poles m and their locations.

I take this opportunity to express my gratitude to M. M. Dzhrbashyan for discussions and remarks that proved very substantial in carrying out the present work.

Mathematical Institute named after V. A. Steklov
Academy of Sciences of the USSR

Received
5 II 1960

CITED LITERATURE

1. V. S. Videnskii, UMN, 9, No. 2 (60), 212 (1954).

Note: Figure translations are in progress. See original paper for figures.