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Abstract

Full Text

Mathematics

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ON THE ASYMPTOTIC BEHAVIOR OF EIGENFUNCTIONS OF THE STURM-LIOUVILLE PROBLEM COMPUTED BY PERTURBATION THEORY

(Presented by Academician A. A. Dorodnitsyn, 16 XII 1959)

We consider the differential operator

$$Au = -\frac{d^2u}{dx^2} + pu, \quad u(0) = u(\pi) = 0.$$

Suppose that $p^{(k)}(x)$ satisfies a Lipschitz condition of order 1 and that all odd derivatives vanish at the ends of the interval $(0, \pi)$ (therefore

$$\int_0^\pi p \cos nx \, dx = O\left(\frac{1}{n^{k+1}}\right),$$

moreover, without loss of generality, one may assume that

$$\int_0^\pi p(x) \, dx = 0,$$

since if a constant is subtracted from the function $p(x)$, then all eigenvalues are decreased by this constant, while the eigenfunctions do not change.

In the work of L. A. Dikii ⁽¹⁾ it was shown that if the operator Au is regarded as the operator $-d^2u/dx^2$ perturbed by the operator pu , then the approximations to the eigenvalues λ_n , computed by perturbation theory up to order k , give the asymptotics for large n up to order $2k$. Below it is shown that the corresponding fact also holds for the eigenfunctions.

Let λ_n be the eigenvalues of the operator A , and let φ_n be the corresponding eigenfunctions. Then, by perturbation theory, λ_n and φ_n are computed as follows:

$$\lambda_n = n^2 + \mu_n^{(1)} + \mu_n^{(2)} + \dots,$$

where

$$\mu_n^{(1)} = \frac{2}{\pi} \int_0^\pi p \sin^2 nx \, dx, \quad \mu_n^{(2)} = \frac{4}{\pi^2} \sum_{m \neq n} \frac{\left(\int_0^\pi p \sin nx \sin mx \, dx \right)^2}{n^2 - m^2};$$

$$\varphi_n(x) = \sqrt{\frac{2}{\pi}} \sin nx + \psi_n^{(1)} + \psi_n^{(2)} + \dots,$$

where

$$\psi_n^{(1)} = \sqrt{\frac{2}{\pi}} \frac{2}{\pi} \sum_{m \neq n} \frac{\int_0^\pi p \sin nx \sin mx \, dx}{n^2 - m^2} \sin mx,$$

$$\psi_n^{(2)} = \frac{2}{\pi} \sum_{m \neq n} \frac{\int_0^\pi (p\psi_n^{(1)} - \mu_n^{(1)}\psi_n^{(1)}) \sin mx \, dx}{n^2 - m^2} \sin mx - \frac{1}{2} \sqrt{\frac{2}{\pi}} \int_0^\pi (\psi_n^{(1)})^2 \, dx \sin nx.$$

Theorem. For the normalized eigenfunctions of the operator A , the following formulas hold (uniformly in x):

$$\varphi_n(x) - \sqrt{\frac{2}{\pi}} \sin nx - \psi_n^{(1)} = O\left(\frac{1}{n^2}\right); \tag{1}$$

$$\varphi_n(x) - \sqrt{\frac{2}{\pi}} \sin nx - \psi_n^{(1)} - \psi_n^{(2)} = O\left(\frac{1}{n^3}\right); \tag{2}$$

.....

$$\varphi_n(x) - \sqrt{\frac{2}{\pi}} \sin nx - \psi_n^{(1)} - \psi_n^{(2)} - \dots - \psi_n^{(k)} = O\left(\frac{1}{n^{k+1}}\right). \tag{k}$$

Proof. It is easy to verify that

$$\sum_{m \neq n} \frac{\int_0^\pi p \sin nx \sin mx \, dx}{n^2 - m^2} \sin mx = \frac{1}{4n} \sum_{m \neq n} \frac{\int_0^\pi p \cos(n - m)x \, dx}{n - m} \sin mx +$$

$$+\frac{1}{4n} \sum_{m \neq n} \frac{\int_0^\pi p \cos(n-m)x dx}{n+m} \sin mx + O\left(\frac{1}{n^{k+1}}\right) = S_1 + S_2 + O\left(\frac{1}{n^{k+1}}\right).$$

The first term

$$\begin{aligned} S_1 &= \frac{1}{4n} \sum_{m \neq n} \frac{\int_0^\pi p \cos(n-m)x dx}{n-m} \sin mx = \\ &= -\frac{1}{2n} \sum_{m=1}^{n-1} \frac{\int_0^\pi p \cos mx dx}{m} \sin mx \cos nx + O\left(\frac{1}{n^{k+2}}\right) = \\ &= -\frac{1}{2n} \sum_{m=1}^{\infty} \left(\int_0^\pi \int_0^x p(t) dt \sin mx dx \right) \sin mx \cos nx + O\left(\frac{1}{n^{k+2}}\right) = \\ &= -\frac{\pi}{4n} \int_0^x p(t) dt \cos nx + O\left(\frac{1}{n^{k+2}}\right). \end{aligned}$$

Now let us find the asymptotics of the second term S_2 . For $k = 1$, obviously, $S_2 = O\left(\frac{1}{n^2}\right)$. For $k > 1$,

$$\begin{aligned} S_2 &= \frac{1}{4n} \sum_{m \neq n} \frac{\int_0^\pi p \cos(n-m)x dx}{n+m} \sin mx = \sum_{m=1}^{n-1} \frac{\int_0^\pi p \cos mx dx}{4n^2 - m^2} \cos mx \sin nx - \\ &\quad - \frac{1}{2n} \sum_{m=1}^{n-1} \frac{m \int_0^\pi p \cos mx dx}{4n^2 - m^2} \sin mx \cos nx + O\left(\frac{1}{n^{k+2}}\right) = \\ &= \frac{\pi}{8n^2} \left[p(x) - \frac{1}{4n^2} p''(x) + \dots + \left(-\frac{1}{4n^2}\right)^{\left[\frac{k-1}{2}\right]} p^{2\left[\frac{k-1}{2}\right]}(x) \right] \sin nx + \\ &\quad + \frac{\pi}{16n^3} \left[p'(x) - \frac{1}{4n^2} p'''(x) + \dots + \left(-\frac{1}{4n^2}\right)^{\left[\frac{k-1}{2}\right]} p^{2\left[\frac{k-1}{2}\right]+1}(x) \right] \cos nx + O\left(\frac{1}{n^{k+1}}\right). \end{aligned}$$

We have, therefore,

$$\psi_n^{(1)} = -\sqrt{\frac{2}{\pi}} \frac{1}{2n} \int_0^x p(t) dt \cos nx + O\left(\frac{1}{n^2}\right) \quad \text{for } k = 1;$$

$$\psi_n^{(1)} = -\sqrt{\frac{2}{\pi}} \frac{1}{2n} \int_0^x p(t) dt \cos nx + \sqrt{\frac{2}{\pi}} \frac{1}{4n^2} p(x) \sin nx + O\left(\frac{1}{n^3}\right) \quad \text{for } k = 2.$$

Hence, using the known asymptotic formula for the eigenfunctions

$$\varphi_n(x) = \sqrt{\frac{2}{\pi}} \sin nx + \sqrt{\frac{2}{\pi}} \frac{1}{n} a_1(x) \cos nx + O\left(\frac{1}{n^2}\right),$$

where

$$a_1(x) = -\frac{1}{2} \int_0^x p(t) dt,$$

we obtain (1).

Let us proceed to the proof of (2). If one takes into account that

$$\mu_n^{(1)} = \frac{2}{\pi} \int_0^\pi p \sin^2 nx dx = O\left(\frac{1}{n^3}\right),$$

then the first term of $\psi_n^{(2)}$ is simplified:

$$\begin{aligned} & \frac{2}{\pi} \sum_{m \neq n} \frac{\int_0^\pi (p\psi_n^{(1)} - \mu_n^{(1)}\psi_n^{(1)}) \sin mx dx}{n^2 - m^2} \sin mx = \\ & = \sqrt{\frac{2}{\pi}} \frac{2}{\pi} \frac{1}{n} \sum_{m \neq n} \frac{\int_0^\pi p a_1 \cos nx \sin mx dx}{n^2 - m^2} \sin mx + O\left(\frac{1}{n^3}\right). \end{aligned}$$

Further, it is not difficult to verify the validity of the following transformations:

$$\sum_{m \neq n} \frac{\int_0^\pi p a_1 \cos nx \sin mx dx}{n^2 - m^2} \sin mx =$$

$$\begin{aligned}
 &= -\frac{1}{2n} \sum_{m=1}^{n-1} \frac{\int_0^\pi p a_1 \sin mx \, dx}{m} \cos mx \sin nx + O\left(\frac{1}{n^2}\right) = \\
 &= -\frac{1}{2n} \sum_{m=1}^{\infty} \frac{\int_0^\pi p a_1 \sin mx \, dx}{m} \cos mx \sin nx + O\left(\frac{1}{n^2}\right) = \\
 &= -\frac{1}{8n} \sum_{m=1}^{\infty} \left[\int_0^\pi \left(\int_0^x p(t) \, dt \right)^2 \cos mx \, dx \right] \cos mx \sin nx + O\left(\frac{1}{n^2}\right).
 \end{aligned}$$

The second term of $\psi_n^{(2)}$ is easily computed:

$$-\frac{1}{2} \sqrt{\frac{2}{\pi}} \int_0^\pi \psi_n^{(1)2} \, dx \sin nx = -\frac{1}{8\pi} \sqrt{\frac{2}{\pi}} \frac{1}{n^2} \int_0^\pi \left(\int_0^x p(t) \, dt \right)^2 \, dx \sin nx + O\left(\frac{1}{n^3}\right).$$

Thus, the second perturbation term $\psi_n^{(2)}$ has the asymptotic form

$$\begin{aligned}
 \psi_n^{(2)} &= -\frac{1}{4\pi} \sqrt{\frac{2}{\pi}} \frac{1}{n^2} \sum_{m=1}^{\infty} \left[\int_0^\pi \left(\int_0^x p(t) \, dt \right)^2 \cos mx \, dx \right] \cos mx \sin nx \\
 &\quad - \frac{1}{8\pi} \sqrt{\frac{2}{\pi}} \frac{1}{n^2} \int_0^\pi \left(\int_0^x p(t) \, dt \right)^2 \, dx \sin nx + O\left(\frac{1}{n^3}\right) \\
 &= -\sqrt{\frac{2}{\pi}} \frac{1}{8n^2} \left(\int_0^x p(t) \, dt \right)^2 \sin nx + O\left(\frac{1}{n^3}\right).
 \end{aligned}$$

Thus, we obtain

$$\psi_n^{(1)} + \psi_n^{(2)} = \sqrt{\frac{2}{\pi}} \frac{1}{n} a_1(x) \cos nx + \sqrt{\frac{2}{\pi}} \frac{1}{n^2} \left[\frac{1}{4} p(x) - \frac{1}{8} \left(\int_0^x p(t) \, dt \right)^2 \right] \sin nx + O\left(\frac{1}{n^3}\right).$$

According to the result of L. A. Dikii ⁽²⁾,

$$\varphi_n(x) = \sqrt{\frac{2}{\pi}} \sin nx \left(1 + \frac{a_2(x)}{n^2} \right) + \sqrt{\frac{2}{\pi}} \frac{1}{n} a_1(x) \cos nx + O\left(\frac{1}{n^3}\right),$$

where

$$a_2(x) = \frac{1}{4}p(x) - \frac{1}{8} \left(\int_0^x p(t) dt \right)^2,$$

we conclude that

$$\varphi_n - \psi_n^{(1)} - \psi_n^{(2)} = O\left(\frac{1}{n^3}\right).$$

Approximations of higher orders are considered analogously.

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CITED LITERATURE

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2. L. A. Dikii, *Izv. AN SSSR, ser. matem.*, **19**, 4, 187 (1955).

Note: Figure translations are in progress. See original paper for figures.

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