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Abstract

Full Text

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The Dependence Between Curvature and the Topological Structure of Riemannian Spaces of Even Dimension

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Rauch (^{1}) proved the following theorem:

Theorem R. *Let M be a complete n-dimensional Riemannian space. If the curvature K of the space M at every point and in every two-dimensional direction satisfies the inequality $0 < hL < K \leq L$, where $h \approx 0.74$ is the solution of the equation $\sin \pi\sqrt{h} = \sqrt{h}/2$, then M is homeomorphic either to a sphere or to an elliptic space.*

Klingenberg (^{2}) for even n strengthened Rauch' s theorem, proving the same assertion for $h \approx 0.54$ (h is the solution of the equation $\sin \pi\sqrt{h} = \sqrt{h}$). In the present paper, for even n, the value of the constant h in Theorem R is lowered to $1/4$. In (^{3}) it is shown that this value of h cannot be decreased further.

Thus, for even dimensions, the problem posed by Rauch may be regarded as completely solved. Let us also note that the assertion of the theorem of Bochner-Yano ((^{4}), p. 67, Theorem 5.1) for even dimension is also covered by our theorem.

We shall prove two lemmas, which are also of independent interest. Consider, in a Riemannian space, triangles composed of two shortest arcs and one geodesic line. If the sum of the lengths of any two sides of a triangle is greater than the third, we shall call such a triangle **proper**.

Lemma 1. *If in a complete Riemannian space M the curvature K at every point and in every two-dimensional direction is not less than H, and if the perimeter of a proper triangle ABC, composed of shortest arcs AB, AC and a geodesic BC, is equal to $2\pi/\sqrt{H}$, then the angles B and C are equal to π.*

Proof. Divide the geodesic BC by points E_p $(p = 0, \dots, s)$ $(E_0 = B,\ E_s = C)$ in such a way that the segments $E_{i-1}E_i$ $(i = 1, \dots, s)$ are shortest arcs. In the space M construct the triangles $AE_{i-1}E_i$, and on the sphere S_H of radius $1/\sqrt{H}$ construct the triangles $(AE_{i-1}E_i)_H$ with the same side lengths as the triangles $AE_{i-1}E_i$. Placing next to one another, along the equal side AE_i, the triangles $(AE_{i-1}E_i)_H$ and $(AE_{iE_{i+1}})_H$, we obtain on the sphere S_H a polygon $A'E'_0 \dots E'_i \dots E'_s$. Relying on the theorem on the angles of triangles composed of shortest arcs ((^{5}), p. 127), it is not difficult to show that all the angles of this polygon are not greater than π. But since the perimeter of the polygon $A'E'_0 \ldots E'_i \ldots E'_s$ is equal to $2\pi/\sqrt{H}$, it is either a great circle or a digon; in both cases the angles B and C are equal to π. Applying again the theorem on the angles of triangles in Riemannian spaces ((^{5}), p. 127), we obtain the proof of Lemma 1.

Lemma 2. *Under the conditions of Lemma 1, the perimeter of a proper triangle does not exceed* $2\pi/\sqrt{H}$.

Proof. Suppose the contrary. Let the perimeter of the regular triangle ABC be equal to $2a$, $a > \pi/\sqrt{H}$. Take on the side BC a point D such that the triangle ACD is regular and such that its perimeter, equal to $2b$, is less than $2a$, but not less than $2\pi/\sqrt{H}$. Since π^2/b^2 is not greater than H, Lemma 1 can be applied to the triangle ACD. Hence we obtain that the angle D is equal to π, but then the shortest path AD will go along the line ABD, and, consequently, the perimeters of the triangles ACD and ACB coincide, contrary to the choice of the point D. The contradiction obtained proves Lemma 2.

For what follows we shall need a theorem proved in (^{2}).

Theorem K. *If the curvature K of a compact simply connected Riemannian space is, at every point and in every two-dimensional direction, strictly positive and not greater than L, then every geodesic segment of length $\pi/\sqrt{L}$ is a shortest path.*

Now we can prove our theorem:

Theorem. *Let M be a complete Riemannian space of even dimension. If the curvature K of the space M at every point and in every two-dimensional direction satisfies the inequality $0 < {}^1/_4\,L < K \leq L$, then M is homeomorphic either to a sphere or to an elliptic space.*

Proof. By the conditions of the theorem, the space M is compact and either simply connected itself, or its two-sheeted covering is simply connected (^{6}). In the first case we shall consider M itself, and in the second, its two-sheeted covering.

Thus, we shall always consider a compact simply connected Riemannian space.

Denote by ${}^1/_4 H$ the minimum of the curvature of the space M; $H > L$, $H < 4L$. Choose the number ρ so that $\rho < \pi/\sqrt{L}$ and $\rho > \pi/\sqrt{H}$. On some geodesic lay off a segment AB of length 2ρ. From Theorem K and from the choice of the number ρ it follows that the points A and B are distinct.

α) Consider a segment AE of a geodesic q with origin at the point A and of length ρ. By Lemma 2, the distance from the point E to B does not exceed $\rho = AE$; therefore on the segment AE there exists a point $D(q)$ equidistant from A and B. It is not difficult to show that the point $D(q)$ is unique. An analogous assertion holds for geodesics passing through the point B.

β) Denote by $C(A)$, respectively $C(B)$, respectively $C(A, B)$, the set of points P of the space M whose distances to the points A and B satisfy the relations $PA < PB$, respectively $PB < PA$, respectively $PA = PB$. From the assertion of item α) and from Theorem K it follows that any point $P \in C(A)$ lies inside the segment $AD(q)$ of a geodesic q with origin at the point A and length not exceeding ρ; consequently, every point of the set $C(A)$ is joined with A by a unique shortest path.

An analogous assertion also holds for points of the set $C(B)$.

γ) We now define a homeomorphic mapping $\varphi(P)$ of the space M onto the sphere S_R of radius R. Let A' be an arbitrary point of S_R; by B' denote the point diametrically opposite to A'. At the points A and A' construct orthonormal frames T and T'. Put the vectors of the frames T and T' in one-to-one correspondence. To each geodesic q of the space M passing through the point A put in correspondence the geodesic q' of the sphere S_2, passing through the point A' and making the same angles with the vectors of the frame T' as the geodesic q makes with the vectors of the frame T. Denote by $D'(q')$ the point of the geodesic q' lying in the equatorial plane. The segment $AD(q)$ of the geodesic q we map onto the segment $A'D'(q')$ of the geodesic q', assigning to the point $P \in AD(q)$ the point $P' \in A'D'(q')$, at a distance from the point A' equal to $\pi R \cdot AP/2AD(q)$. Further, to the shortest path $D(q)B$ we put in correspondence the shortest path $D'(q')B'$, and to the point $P \in D(q)B$—the point $P' \in D'(q')B'$, at a distance

from B' at a distance $\pi R \cdot PB/2D(q)B$. From item β), the choice of the number ρ, and Theorem K it follows that the mapping $\varphi(P)$ thus obtained is a homeomorphism of the space M onto the sphere S_R. The theorem is proved.

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Note: Figure translations are in progress. See original paper for figures.