MATHEMATICS

1960

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Abstract

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MATHEMATICS

Yu. I. ZHURAVLEV

ON THE IMPOSSIBILITY OF CONSTRUCTING MINIMAL DISJUNCTIVE NORMAL FORMS OF FUNCTIONS OF THE ALGEBRA OF LOGIC IN ONE CLASS OF ALGORITHMS

(Presented by Academician S. L. Sobolev on 12 I 1960)

Algorithms for simplifying disjunctive normal forms (d.n.f.) make it possible to distinguish in $\mathfrak{N}$ certain conjunctions that do not enter into $\mathfrak{N}_{\Sigma M}$, and to mark certain conjunctions that do enter into $\mathfrak{N}_{\Sigma M}$. Generally speaking, in $\mathfrak{N}$ there exist conjunctions for which it is not possible to obtain information about whether they do or do not enter into $\mathfrak{N}_{\Sigma M}$. It is natural to pose the question whether there exists a sufficiently effective algorithm that makes it possible, for every pair $(\mathfrak{A}, \mathfrak{N})$, where $\mathfrak{N}$ is a reduced d.n.f. and $\mathfrak{A}$ is a conjunction from $\mathfrak{N}$, to establish whether $\mathfrak{A}$ is contained in $\mathfrak{N}_{\Sigma M}$ or is not contained. Formally, this problem admits the following formulation*:

Problem M. Indicate an algorithm A, generated by a function φ, such that:
1) if φ is monotone, then the image of an arbitrary reduced d.n.f. $\mathfrak{N}$ in the algorithm A contains no conjunctions with the mark (0);
2) if φ is nonmonotone, at least one of the images of an arbitrary reduced d.n.f. $\mathfrak{N}$ in the algorithm A contains no conjunctions with the mark (0).

It turns out that, in the class of algorithms that do not use, in one form or another, a significant search, problem M is unsolvable. The aim of the present note is to prove this fact.

To every algorithm A and d.n.f. $\mathfrak{N}$ one can assign a sequence $\mathfrak{N}, \mathfrak{N}_1, \dots, \mathfrak{N}_\alpha$ such that:
1) A successively transforms $\mathfrak{N}$ into $\mathfrak{N}_1, \dots, \mathfrak{N}_{\alpha-1}$ into $\mathfrak{N}_\alpha$;
2) neighboring terms of the sequence differ in their marks over exactly one conjunction;
3) if the algorithm A is generated by a monotone function φ, then $\mathfrak{N}_\alpha$ is the image of the d.n.f. $\mathfrak{N}$ in the algorithm A;
4) if the algorithm A is generated by a nonmonotone function φ, then $\mathfrak{N}_1, \dots, \mathfrak{N}_\alpha$ are all the images of the d.n.f. $\mathfrak{N}$ in the algorithm A.

We shall call the algorithm A **admissible** if to $\mathfrak{A}$ and an arbitrary admissible d.n.f. $\mathfrak{N}$ there corresponds a sequence $\mathfrak{N}_1, \dots, \mathfrak{N}_\alpha$ of admissible d.n.f.' s. We shall call a function φ **admissible** if all simplification algorithms generated by the function φ are admissible. In what follows we shall consider only admissible functions.

Let us introduce the inductive notion of the principal neighborhood of order k of a conjunction $\mathfrak{A}^{(j)}$ in a d.n.f. $\mathfrak{N}$. The **principal neighborhood of first order** $S_1(\mathfrak{A}^{(j)}, \mathfrak{N})$ of the conjunction $\mathfrak{A}^{(j)}$ in the d.n.f. $\mathfrak{N}$ will be called the set of all such conjunctions from $\mathfrak{N}$ that the intervals corresponding to them (2) have a nonempty intersection with the interval $N_{\mathfrak{A}}$. Suppose that the principal neighborhood of order $(k-1)$ of the conjunction $\mathfrak{A}^j$ in the d.n.f. $\mathfrak{N}$ has been defined. The **principal neighborhood** $S_k(\mathfrak{A}^{(j)}, \mathfrak{N})$ **of order** k **of the conjunction** $\mathfrak{A}^{(j)}$ in the d.n.f. $\mathfrak{N}$

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* In the present note we shall use the terminology and notation of our note (1).

we shall call the totality of all conjunctions $\mathfrak{A}_k^{(l)}$ from $\mathfrak{N}$ for which one of the following conditions is fulfilled: 1) the interval corresponding to the conjunction $\mathfrak{A}_k^{(j)}$ has a nonempty intersection with the interval corresponding to a conjunction from $S_{k-1}(\mathfrak{A}^{(j)}, \mathfrak{N})$; 2) the interval corresponding to the conjunction $\mathfrak{A}_k^{(l)}$ is contained in the sum of intervals, to each of which there corresponds a conjunction from $\mathfrak{N}$ satisfying condition 1).

Obviously,

$$M[S_1(\mathfrak{A}^{(j)}, \mathfrak{N})] \subseteq M[S_2(\mathfrak{A}^{(j)}, \mathfrak{N})] \subseteq \cdots \subseteq M[S_k(\mathfrak{A}^{(j)}, \mathfrak{N})].$$

Let the domain of definition of the function φ be a system of neighborhoods, and suppose that with each pair $(\mathfrak{A}^{(j)}, \mathfrak{N})$, where $\mathfrak{A}^{(j)} \subseteq \mathfrak{N}$ and $\mathfrak{N}$ is an admissible d.n.f., there is associated a neighborhood $S(\mathfrak{A}^{(j)}, \mathfrak{N})$. The function φ has index k if: 1) for every pair $(\mathfrak{A}^{(j)}, \mathfrak{N})$, where $\mathfrak{A}^{(j)} \subseteq \mathfrak{N}$, the relation

$$M[S(\mathfrak{A}^{(j)}, \mathfrak{N})] \subseteq M[S_k(\mathfrak{A}^{(j)}, \mathfrak{N})]$$

holds; 2) there exists a pair $(\mathfrak{A}^{(j)}, \mathfrak{N})$, where $\mathfrak{A}^{(j)} \subseteq \mathfrak{N}$, such that

$$M[S(\mathfrak{A}^{(j)}, \mathfrak{N})] \nsubseteq M[S_{k-1}(\mathfrak{A}^{(j)}, \mathfrak{N})].$$

We shall say that a simplification algorithm has index k if it is generated by a function φ of index k. Let D_k be the set of all algorithms of index k. An algorithm A has finite index if

$$A \subseteq \bigcup_{k=1}^{\infty} D_k.$$

We shall call a d.n.f. $\mathfrak{N}$ **absolutely irreducible in the algorithm** A if there exists a unique image $\mathfrak{N}_1$ of the d.n.f. $\mathfrak{N}$ in the algorithm A, and the d.n.f.' s $\mathfrak{N}$ and $\mathfrak{N}_1$ are equal in information.

Theorem. *For each of the algorithms A_k of index k there is a reduced d.n.f. $\mathfrak{N}^{(2)}_{\varphi_k}$, composed of unmarked conjunctions and absolutely irreducible in the algorithm A_k.*

Proof. Consider the function of the algebra of logic $f(x_1, \dots, x_n)$, $n \geq 2k+6$, equal to one on the tuples

$$(0,0,\dots,0,0), \quad (0,0,\dots,0,1), \quad (0,0,\dots,0,1,1),\dots, \quad (0,1,1,\dots,1,1),$$

$$(1,1,\dots,1,1), \quad (1,1,\dots,1,0),\dots, \quad \dots,(1,0,\dots,0,0).$$

Write the reduced d.n.f. $\mathfrak{N}_{f_k}$ of the function f_k:

$$\mathfrak{N}_{f_k} = \bar{x}_1 \cdot \bar{x}_2 \cdot \dots \cdot \bar{x}_{n-1} \vee \bar{x}_1 \cdot \bar{x}_2 \cdot \dots \cdot \bar{x}_{n-2} \cdot x_n \vee$$

$$\vee \bar{x}_1 \cdot \bar{x}_2 \cdot \dots \cdot \bar{x}_{n-3} \cdot x_{n-1} \cdot x_n \vee \dots \vee x_2 \cdot x_3 \cdot \dots \cdot x_{n-1} \cdot x_n \vee$$

$$\vee x_1 \cdot x_2 \cdot \dots \cdot x_{n-2} \cdot x_{n-1} \vee x_1 \cdot x_2 \cdot \dots \cdot x_{n-2} \cdot \bar{x}_n \vee \dots \vee \bar{x}_2 \cdot \bar{x}_3 \cdot \dots \cdot \bar{x}_n.$$

Let the algorithm A be generated by a function φ of index k. We shall show that on the neighborhood $S(\mathfrak{A}, \mathfrak{N}_{f_k})$, where $\mathfrak{A}$ is an arbitrary conjunction from $\mathfrak{N}_{f_k}$ and $S(\mathfrak{A}, \mathfrak{N}_{f_k})$ is a neighborhood belonging to the domain of definition of the function φ, the value of φ is equal to (0). For definiteness consider the conjunction

$$\mathfrak{A}_1 = \bar{x}_1 \cdot \bar{x}_2 \cdot \dots \cdot \bar{x}_{n-1}.$$

Write out its principal neighborhood $S_k(\mathfrak{A}_1, \mathfrak{N}_{f_k})$:

$$S_k(\mathfrak{A}_1, \mathfrak{N}_{f_k}) = \{\bar{x}_1 \cdot \bar{x}_2 \cdot \dots \cdot \bar{x}_{n-1},\ \bar{x}_1 \cdot \bar{x}_2 \cdot \dots \cdot \bar{x}_{n-2} \cdot x_n,$$

$$\bar{x}_1 \cdot \bar{x}_2 \cdot \dots \cdot \bar{x}_{n-3} \cdot x_{n-1} \cdot x_n, \dots,\ \bar{x}_1 \cdot \bar{x}_2 \cdot \dots \cdot \bar{x}_{n-k-1} \cdot x_{n-k+1} \cdot \dots \cdot x_n;$$

$$\bar{x}_2 \cdot \bar{x}_3 \cdot \dots \cdot \bar{x}_n,\ x_1 \cdot \bar{x}_3 \cdot \dots \cdot \bar{x}_n,\ x_1 \cdot x_2 \cdot x_4 \cdot \dots \cdot \bar{x}_n, \dots,\ x_1 \cdot x_2 \cdot \dots \cdot x_{n-1} \cdot \bar{x}_{k+1} \cdot \dots \cdot \bar{x}_n\}.$$

From the definition of an algorithm of index k it follows that

$$M[S(\mathfrak{A}_1, \mathfrak{N}_{f_k})] \subseteq M[S_k(\mathfrak{A}_1, \mathfrak{N}_{f_k})].$$

Note that

$$\mathfrak{A}_1 \subseteq (\mathfrak{N}_{f_k})_{\Sigma M}.$$

Write a reduced d.n.f. $\mathfrak{N}_{\psi_k}$ such that

$$S(\mathfrak{A}_1, \mathfrak{N}_{\psi_k}) = S(\mathfrak{A}_1, \mathfrak{N}_{f_k}) \quad \text{and} \quad \mathfrak{A}_1 \nsubseteq (\mathfrak{N}_{\psi_k})_{\Sigma M}.$$

If k is even, then

$$\mathfrak{N}_{\psi_k} = \bar{x}_1 \cdot \bar{x}_2 \cdot \dots \cdot \bar{x}_{n-1} \vee \bar{x}_1 \cdot \bar{x}_2 \cdot \dots \cdot \bar{x}_{n-2} \cdot x_n \vee \bar{x}_1 \cdot \bar{x}_2 \cdot \dots \cdot \bar{x}_{n-3} \cdot x_{n-1} \cdot x_n \vee$$

$$\vee \bar{x}_1 \cdot \bar{x}_2 \cdot \dots \cdot \bar{x}_{n-k-1} \cdot x_{n-k+1} \cdot \dots \cdot x_n \vee \bar{x}_2 \cdot \bar{x}_3 \cdot \dots \cdot \bar{x}_n \vee x_1 \cdot \bar{x}_3 \cdot \dots \cdot \bar{x}_n \vee$$

$$\vee x_1 \cdot x_2 \cdot x_4 \cdot \dots \cdot \bar{x}_n \vee \dots \vee x_1 \cdot x_2 \cdot \dots \cdot x_{k-1} \cdot \bar{x}_k \cdot x_{k+1} \cdot \dots \cdot x_n \vee \bar{x}_1 \cdot \bar{x}_2 \cdot \dots$$

$$\ldots \cdot x_{n-k+1} \cdot \bar{x}_{n-k} \cdot x_{n-k+2} \cdot \ldots \cdot x_n \vee x_1 \cdot x_2 \cdot \ldots \cdot x_{k-1} \cdot x_k \cdot \bar{x}_{k+2} \cdot \ldots \cdot \bar{x}_n.$$

If k is odd, then

$$\mathfrak{N}_{\psi_k} = \bigvee_{i=1}^{2k+1} \mathfrak{A}_i,$$

where $\mathfrak{A}_i$ are all the conjunctions entering into

$$M[S(\mathfrak{A}_1, \mathfrak{N}_{f_k})].$$

It is not difficult to see that both for even k and for odd k

$$\mathfrak{A}_1 \not\subseteq (\mathfrak{N}_{\psi_k})_{\Sigma M}.$$

The neighborhoods belonging to the domain of definition of φ possess

the following property: if $M[S(\mathfrak{A}, \mathfrak{N})] \subset M(\mathfrak{N}_1)$ and $M(\mathfrak{N}_1) \subset M(\mathfrak{N})$, then $M[S(\mathfrak{A}, \mathfrak{N})] = M[S(\mathfrak{A}, \mathfrak{N}_1)]$ (1). Obviously, $M(\mathfrak{N}_{\psi_k}) \subseteq M(\mathfrak{N}_{f_k})$ and $M[S(\mathfrak{A}_1, \mathfrak{N}_k)] \subseteq M(\mathfrak{N}_{\psi_k})$. Therefore

$$M[S(\mathfrak{A}_1, \mathfrak{N}_{f_k})] = M[S(\mathfrak{A}_1, \mathfrak{N}_{\psi_k})].$$

Since all conjunctions in $S(\mathfrak{A}_1, \mathfrak{N}_{f_k})$ are unmarked, $S(\mathfrak{A}_1, \mathfrak{N}_{f_k})$ and $S(\mathfrak{A}_1, \mathfrak{N}_{\psi_k})$ are equal in information, and

$$\varphi[S(\mathfrak{A}_1, \mathfrak{N}_{f_k})] = \varphi[S(\mathfrak{A}_1, \mathfrak{N}_{\psi_k})] = \widetilde{\varphi}.$$

From the last equality it follows that

$$\varphi[S(\mathfrak{A}_1, \mathfrak{N}_{f_k})] = (0).$$

Indeed, if $\widetilde{\varphi} \neq (0)$, then the function φ generating the algorithm A is not admissible.

The theorem is proved. We have proved the unsolvability of problem M in the class of algorithms with finite index.

Moscow State University
named after M. V. Lomonosov

Received
7 I 1960

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Note: Figure translations are in progress. See original paper for figures.

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