MATHEMATICS

O. A. LADYZHENSKAYA and N. N. URALTSEVA

1960

SovietRxiv

Abstract

Full Text

MATHEMATICS

O. A. LADYZHENSKAYA and N. N. URALTSEVA

ON A VARIATIONAL PROBLEM AND QUASILINEAR ELLIPTIC EQUATIONS WITH MANY INDEPENDENT VARIABLES

(Presented by Academician V. I. Smirnov on 10 VI 1960)

We shall present results* obtained by us concerning solutions of the variational problem of finding

$$\inf I(u) = \inf \int_\Omega F(x, u, u_{x_k})\, dx, \qquad x = (x_1, \dots, x_n) \tag{1}$$

under the condition

$$u\big|_S = \varphi(s), \tag{2}$$

solutions of the first boundary-value problem for quasilinear elliptic equations of general form

$$L_1(u) \equiv a_{ij}(x, u, u_{x_k}) u_{x_i x_j} + a(x, u, u_{x_k}) = 0 \tag{3}$$

and elliptic equations "self-adjoint in the principal part,"

$$M_1(u) \equiv \frac{\partial}{\partial x_i}(a_i(x, u, u_{x_k})) + a(x, u, u_{x_k}) = 0. \tag{4}$$

The main theorems stated below are new also for $n = 2$.

Notation. E_k is k-dimensional Euclidean space; Ω is a bounded domain, and Ω' is its strictly interior subdomain; S is the boundary of the domain Ω; $C_{l,\alpha}(\Omega)$, $W^l_m(\Omega)$ are the usually defined classes of functions ([1,2]), $O_l(\Omega)$ is the class of functions $u(x)$, $x \in \Omega$, for which derivatives of order $l-1$ have first differentials, while derivatives up to order l inclusive are bounded on every compact part of Ω. We shall say that the domain Ω satisfies condition (A) if there exist constants $a > 0$ and θ in $(0, 1)$ such that for any ball $K(\rho)$ with center on S and radius $\rho \leq a$ the inequality $\operatorname{mes}[K(\rho) \cap \Omega] \leq (1-\theta) \operatorname{mes} K(\rho)$ holds.

§ 1. **A priori estimates.** As S. N. Bernstein showed ([3]), the following restrictions on the coefficients a_{ij}, a of the quasilinear elliptic equations (3) are necessary in order that there may exist an a priori estimate of $\max_\Omega |u_{x_i}|$ in terms of $\max_\Omega |u|$ in an arbitrary domain Ω:

$$\nu(|u|)(p^2+1)^{m/2} \le a_{ij}(x,u,p_k)p_ip_j \le \mu(|u|)(p^2+1)^{m/2}; \tag{5}$$

$$|a(x,u,p_k)| \le \mu(|u|)(p^2+1)^{m/2}, \qquad p = \left(\sum_{k=1}^{n} p_k^2\right)^{1/2}, \tag{6}$$

---

* Part of these results was reported in the autumn of 1959 at V. I. Smirnov' s seminar in Leningrad and in a survey lecture at a meeting of the Leningrad Mathematical Society, and also in December 1959 at I. G. Petrovsky' s seminar in Moscow.

where $\mu(|u|)$ here and below denotes a positive monotonically increasing function, $\nu(|u|)$ a positive monotonically decreasing function $(|u|)$, and m is some number greater than 1. It is assumed here that each differentiation of the functions $a_{ij}(x,u,p_k)$, $a(x,u,p_k)$, $a_i(x,u,p_k)$, $F(x,u,p_k)$ with respect to p_k lowers their order of growth with respect to p by at least 1, while differentiations with respect to x_k and u do not increase them. We shall call these conditions (B). We shall say that the equation is uniformly elliptic if, for $\sum \xi_i^2 = 1$,

$$\nu(|u|)(p^2+1)^{m/2-1} \le a_{ij}(x,u,p_k)\xi_i\xi_j \le \mu(|u|)(p^2+1)^{m/2-1}. \tag{7}$$

Theorem 1. Let $u(x)$ be a solution of equation (3), belonging to the class $O_3(\Omega) \cap C_1(\overline{\Omega})$ and satisfying condition (2), and let $a_{ij}(x,u,p_k)$, $a(x,u,p_k) \in O_1(\Omega \times E_1 \times E_n)$, and suppose that conditions (B) and (7) are fulfilled for them. Then $\max_\Omega |u_{x_i}|$ is estimated in terms of $\max_\Omega |u|$ and $|\varphi|_{C_{2,0}(S)}$, if the oscillation of $u(x)$ in Ω is small* and the boundary S belongs to $C_{2,0}$.

Theorem 2. Suppose the conditions of Theorem 1 are fulfilled, except for the assumptions on S and φ. Then for any $\Omega' \subset \Omega$, $\max_{\Omega'} |u_{x_i}|$ is estimated in terms of $\max_\Omega |u|$.

Theorem 3. Suppose the conditions of Theorem 1 are fulfilled, except for the assumption on the smallness of the oscillation of $u(x)$ in Ω, and suppose that there exists a sufficiently large $R > 0$ such that, for $p \ge R$, $|u| \le M$, the quantities

$$\frac{1}{p^{m-2}}\left|\frac{\partial a_{ij}(x,u,p_k)}{\partial u}\right|, \qquad \frac{1}{p^m}\frac{\partial a(x,u,p_k)}{\partial u} \tag{8}$$

do not exceed a certain number $\varepsilon > 0$, determined by the data of the problem. Then, for the solution u of problem (3), (2), $\max_\Omega |u_{x_i}|$ can be estimated in terms of $\max_\Omega |u|$ and $|\varphi|_{C_{2,0}(S)}$.

Suppose that for equations (4) conditions (B) and (7) are fulfilled, where $m-1$ is the order of growth of $a_i(x,u,p_k)$ with respect to p. We single out from them an important class of quasilinear equations for which, for large p, one must have

$$a_i(x,u,p_k)p_i \geq \nu(|u|)p^m, \qquad p \gg 1. \tag{9}$$

Let us note that, for linear equations and for equations (3) not containing u_{x_i} in a_{ij}, condition (9) is a consequence of the ellipticity condition. For the Euler equation it is a consequence of the natural assumptions, indicated below, concerning F.

Theorem 4. If for (4) only the conditions just enumerated are fulfilled and $a_i(x,u,p_k) \in O_2(\Omega \times E_1 \times E_n)$, $a(x,u,p_k) \in O_1(\Omega \times E_1 \times E_n)$, then for any solution u of equation (4) from $O_3(\Omega)$ the norm $|u|_{C_{1,\alpha}(\Omega')}$, with some $\alpha > 0$, can be estimated in terms of $\max_\Omega |u|$. If, moreover, $\varphi \in C_{2,0}$, $S \in O_2$, then the norm $|u|_{C_{1,\alpha}(\Omega)}$ is estimated in terms of $\max_\Omega |u|$ and $|\varphi|_{C_{2,0}}$.

Theorem 5. Suppose all the assumptions of Theorem 4 are fulfilled, except for inequality (9). Then, if the functions $\partial^2 a_i(x,u,p_k)/\partial p_j \partial u$, $\partial^2 a_i(x,u,p_k)/\partial u^2$, $\partial a(x,u,p_k)/\partial u$ have, with respect to p, orders of growth $m-2-\varepsilon$, $m-1-\varepsilon$, $m-\varepsilon$ ($\varepsilon > 0$), respectively, then the norm $|u|_{C_{1,\alpha}(\Omega')}$ is estimated in terms of $\max_\Omega |u|$. If $\varphi \in C_{2,0}$, $S \in O_2$, then the norm $|u|_{C_{1,\alpha}(\Omega)}$ is estimated in terms of $\max_\Omega |u|$ and $|\varphi|_{C_{2,0}}$.

---

* That is, if $\operatorname{osc}\{u;\Omega\}$ is less than a certain number (which we do not give here), determined by the constants entering into the conditions of the problem.

§ 2. **Existence theorems.** Consider

$$M_\tau(u) \equiv \tau M_1[u] + (1-\tau)M_0(u) = 0, \qquad u|_S = \tau\varphi(s), \tag{10}$$

where

$$M_0(u) = \frac{\partial}{\partial x_i} F^0_{u_{x_i}} - F^0_u, \qquad F^0(x,u,u_{x_k}) = \left(\sum_i u^2_{x_i} + 1\right)^{m/2} + u^2.$$

Theorem 6. Suppose that: 1) for $M_1(u)$ the conditions of Theorem 4 or 5 are satisfied; 2) $a_i(x,u,p_k)$, $a(x,u,p_k)$, as functions of all their arguments, belong to $C_{2,\alpha}$, $C_{1,\alpha}$, respectively; 3) $S \in C_{2,\alpha}$, $\varphi \in C_{2,\alpha}$. Then problem (10) has at least one solution $u(x,\tau)$ for all $\tau \in [0,1]$, if for all possible solutions $\max|u(x,\tau)|$ is uniformly bounded. All solutions lie in the space

$$C^{\Omega}_{2,\alpha}(\overline{\Omega}) \cap C_{3,\alpha}(\Omega).$$

For $n = 2$ the following theorem is also valid:

Theorem 7. Let for equation (3), with $n = 2$, the conditions ( ) and (7) be satisfied, and without loss of generality let us assume $m = 2$. Suppose that instead of (6) the stronger restriction

$$|a(x,u,p_k)| \le \mu(|u|)(p^2+1)^{1-\varepsilon}, \qquad \varepsilon > 0$$

is satisfied. Then the problem

$$L_\tau(u) \equiv \tau L_1(u) + (1-\tau)(\Delta u - u) = 0, \qquad u|_S = \tau\varphi(s)$$

has at least one solution $u(x,\tau)$ in $C_{2,\alpha}(\overline{\Omega}) \cap C_{3,\alpha}(\Omega)$ for all $\tau \in [0,1]$, provided only that, for all possible such solutions $u(x,\tau)$, the moduli are uniformly bounded. The functions a_{ij}, a must belong to $C_{1,\alpha}$, $\varphi \in C_{2,\alpha}$, $S \in C_{2,\alpha}$, and Ω is homeomorphic to a disk.

§ 3. **The variational problem.** Let $F(x,u,p_k)$ have growth order $m > 1$ with respect to p, and let each differentiation with respect to p_k lower the growth order in p by at least 1, while differentiations with respect to x_k and u do not increase it. Suppose, further,*

$$F(x,u,p_k) \ge \nu_1(|u|)p^m;$$

$$F_{p_ip_j}(x,u,p_k)\xi_i\xi_j \ge \nu_2(|u|)(p^2+1)^{(m-2)/2}\sum_i \xi_i^2; \tag{11}$$

$$F_{p_i}(x,u,p_k)p_i \ge \nu_3(|u|)p^m, \qquad p \gg 1.$$

Then, as is not difficult to see, for the Euler equation for $I(u)$ Theorem 4 is valid if $F \in C_3$, and Theorem 6 if $F \in C_{3,\alpha}$. One of the fundamental questions is that of the conditions on F under which every generalized solution of the variational problem possesses certain differential properties.

We have proved the following theorems:

Theorem 8. Let u be a generalized solution from $W^1_m(\Omega)$ of the "conditional" variational problem (1), (2), i.e. the problem to which the condition is added that all comparison functions do not exceed some constant $M \ge \max_S|u|$. It will belong to $C_{0,\alpha}(\Omega)$, if $F \in C_1$ and the conditions

$$\mu(|u|)p^m \geq F_{p_i}(x,u,p_k)p_i \geq \nu(|u|)p^m, \qquad p \gg 1,$$

$$|F_u(x,u,p_k)| \leq \mu(|u|)p^m. \tag{12}$$

are satisfied.

* If F is representable in the form $F(x,u,p_k) = F'(x,u,p_k) + F''(x,u,p_k)$, where F' is a positive homogeneous function of p_k of order m, and $\frac{1}{p^m}F'' \to 0$ as $p \to \infty$, then (11) are consequences of the positivity and convexity of F and F'.)

Under the same conditions on F, every bounded function u from $W_m^1(\Omega)$ for which $\delta I(u) = 0$ belongs to $C_{0,\alpha}(\Omega)$. If, moreover, Ω satisfies condition (A) and $\varphi \in C_1$, then $u \in C_{0,\alpha}(\overline{\Omega})$.

Theorem 9. Under the conditions on F formulated at the beginning of the paragraph, every bounded generalized solution $u(x)$ of the variational problem (1), (2) from the class $W_m^1(\Omega)$ belongs to $C_{k,\alpha}(\Omega)$, if $F \in C_{k,\alpha}$, $k \geqslant 3$, and if

$$\Delta I(u) = I(u+\eta) - I(u) > 0$$

for every sufficiently small local variation $\eta(x)$. If, moreover, $S \in C_{l,\alpha}$, $\varphi \in C_{l,\alpha}$, $2 \leqslant l \leqslant k$, then

$$u \in C_{l,\alpha}(\overline{\Omega}) \cap C_{k,\alpha}(\Omega).$$

For $m = 2$ the condition $\Delta I(u) > 0$ may be omitted.

Finally, we give two lemmas which generalize De Giorgi' s lemma [4].

Let $u \in W_m^1(\Omega)$, $m > 1$. Denote by $A_{k,\rho}$ the set of points of the ball $K(\rho)$ of radius ρ, lying entirely in Ω, for which $u(x) > k$, and by $B_{k,\rho}$ the set of points of $K(\rho)$ for which $u < k$. We shall say that the function u belongs to the class $\mathfrak{B}_m(\Omega; M; \gamma, \delta)$ if $u \in W_m^1(\Omega)$, $|u| \leqslant M$, and for every $K(\rho) \subset \Omega$ the estimate

$$\int_{K(\rho/2)} |\operatorname{grad} u|^m \, dx \leqslant \gamma \rho^{n-m} \tag{13}$$

holds.

In addition, for those k_1 for which

$$\max_{A_{k_1,\rho}} |u(x) - k_1| \leqslant \delta,$$

for any σ from $(0,1)$ the inequality

$$\int_{A_{k_1,\rho-\sigma\rho}} |\operatorname{grad} u|^m\,dx \leqslant \gamma \operatorname{mes} A_{k_1,\rho}\left\{\frac{1}{(\sigma\rho)^m}\max_{A_{k_1,\rho}}|u(x)-k_1|^m+1\right\}, \quad (14)$$

holds; and for k_2 for which

$$\max_{B_{k_2,\rho}}|k_2-u(x)|\leqslant\delta,$$

the inequality

$$\int_{B_{k_2,\rho-\sigma\rho}} |\operatorname{grad} u|^m\,dx \leqslant \gamma \operatorname{mes} B_{k_2,\rho}\left\{\frac{1}{(\sigma\rho)^m}\max_{B_{k_2,\rho}}|k_2-u(x)|^m+1\right\}. \quad (15)$$

holds.

Here M,γ,δ, and m are fixed positive numbers, with $1<m\leqslant n$.

Lemma 1. Let $u\in\mathfrak{B}_m(\Omega;M;\gamma,\delta)$; let x_0 be an arbitrary interior point of Ω; and let ρ_0 be the distance from x_0 to the boundary S. Then for every ball $K(\rho)$ with center at x_0 and radius $\rho\leqslant\rho_0$ the estimate

$$\operatorname{osc}(u,K(\rho))\leqslant C\rho_0^{-\alpha}\rho^\alpha$$

holds, where $C>0$ and some $\alpha\in(0,1)$ are constants for the given class.

Denote by $\mathfrak{B}_m^0(\Omega;M;\gamma,\delta)$ the class of functions u from $\mathfrak{B}_m(\Omega;M;\gamma,\delta)$ satisfying the following requirements: a) $u\in W_m^1(\Omega)$ and $u\equiv 0$ outside Ω; b) if the ball $K(\rho)$ intersects the boundary S, then inequality (13) is preserved, while inequalities (14) and (15) hold not for all the indicated values k_1 and k_2, but for $k_1\geqslant 0$ and $k_2\leqslant 0$, respectively.

Lemma 2. Let $u\in\mathfrak{B}_m^0(\Omega;M;\gamma,\delta)$, and let Ω have property (A). Then for every $K(\rho)$

$$\operatorname{osc}\{u;K(\rho)\}\leqslant C\rho^\alpha$$

with constants C and $\alpha>0$ determined for the given class.

Leningrad State University
named after A. A. Zhdanov

Received
2 VI 1960

REFERENCES

1. K. Miranda, *Equations with Partial Derivatives of Elliptic Type*, IL, 1957.

2. S. L. Sobolev, *Some Applications of Functional Analysis in Mathematical Physics*, L., 1950.

3. S. N. Bernstein, UMN, vol. 8 (1941).

4. E. de Giorgi, Mem. d. Acad. Sc. Torino, s. 3, t. 3° (1957).

Note: Figure translations are in progress. See original paper for figures.