Mathematics

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1960

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Abstract

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ON THE LOCATION OF THE ZEROS OF THE DERIVATIVE OF AN ENTIRE FUNCTION WITH ZEROS CLOSE TO THE REAL AXIS

(Presented by Academician S. N. Bernstein, October 12, 1959)

The classical theorem of Laguerre states:

If a real entire function $f(z)$ has only real zeros and is representable in the form

$$f(z) = e^{-\gamma z^2} P(z), \tag{1}$$

where $\gamma \geqslant 0$, $P(z)$ is an entire function of genus 1, then all derivatives of $f(z)$ also have only real zeros and are likewise representable in the form (1).

In the present note we consider entire functions of the class* A. We establish, in particular, the following theorem, which may be regarded as an analogue of Laguerre' s theorem:

Theorem 1. *If an entire function $f(z)$ belongs to the class A and is representable in the form*

$$f(z) = e^{Q(z)} P(z), \tag{2}$$

where $Q(z)$ is an entire function of exponential type satisfying the condition

$$\int_{-\infty}^{\infty} \frac{\ln^+ |Q(t)|}{1+t^2} \, dt < \infty, \tag{3}$$

and $P(z)$ is an entire function for which

$$\int_{1}^{\infty} \frac{\ln^+ \ln^+ M(t, P)}{t^2} \, dt < \infty, \tag{4}$$

then all derivatives of $f(z)$ also belong to the class A and are likewise representable in the form (2).

1°. Notation and results used. Let $f(z)$ be a function meromorphic in the whole finite plane. Following Nevanlinna ([1]), denote

$$A_{\alpha\beta}(r,f) = \frac{1}{\gamma}\int_1^r \left(\ln^+\left|f\left(te^{i\alpha}\right)\right| + \ln^+\left|f\left(te^{i\beta}\right)\right|\right)\left(\frac{1}{t^{\pi/\gamma}} - \frac{t^{\pi/\gamma}}{r^{2\pi/\gamma}}\right)\frac{dt}{t}$$

$$(0 < \beta - \alpha \leqslant 2\pi, \quad \gamma = \beta - \alpha),$$

* Recall that this class consists of entire functions with zeros $\{a_k\}_{k=1}^{\infty}$ satisfying the condition

$$\sum_{k=1}^{\infty}\left|\operatorname{Im}\left(a_k^{-1}\right)\right| < \infty,$$

the meaning of which is that the zeros of the function "do not recede too rapidly" from the real axis.

$$B_{\alpha\beta}(r,f) = \frac{2}{\gamma r^{\pi/\gamma}}\int_\alpha^\beta \ln^+|f(re^{i\varphi})|\sin\frac{\pi}{\gamma}(\varphi-\alpha)\,d\varphi,$$

$$C_{\alpha\beta}(r,f) = 2\sum_{\substack{1<r_k<r\\ \alpha<\varphi_k<\beta}}\left(\frac{1}{r_k^{\pi/\gamma}} - \frac{r_k^{\pi/\gamma}}{r^{2\pi/\gamma}}\right)\sin\frac{\pi}{\gamma}(\varphi_k-\alpha)$$

($r_k e^{i\varphi_k}$ are the poles of $f(z)$, counted with multiplicities),

$$S_{\alpha\beta}(r,f) = A_{\alpha\beta}(r,f) + B_{\alpha\beta}(r,f) + C_{\alpha\beta}(r,f).$$

Nevanlinna theorem N **([1]).** For any complex a ($\neq\infty$) the relation

$$S_{\alpha\beta}(r,(f-a)^{-1}) = S_{\alpha\beta}(r,f) + O(1)$$

holds.

Theorem A. If the function $f(z)$ satisfies the condition

$$\int_1^\infty (\ln^+ T(t,f))\,t^{-\frac{\pi}{\gamma}-1}\,dt < \infty,$$

then

$$A_{\alpha\beta}(r,f'/f) = O(1), \qquad B_{\alpha\beta}(r,f'/f) = O(1).$$

The proof of this theorem is carried out by a method very close to that which Nevanlinna used to prove his well-known "lemma on the logarithmic derivative" ([2]), pp. 56–61). We note that the estimate of the quantity $B_{\alpha\beta}(r, f'/f)$ is a trivial consequence of this lemma.

2°. In this paragraph we shall prove one theorem on the distribution of the zeros of the derivatives of a certain class of meromorphic functions, and in **3°** we shall establish that theorem 1 is its simple consequence.

We introduce the necessary definitions.

Definition 1. We shall say that the zeros and poles of the meromorphic function $f(z)$ lying inside the angle $\alpha \le \arg z \le \beta$ $(0 < \beta - \alpha \le 2\pi)$ are **very close to the sides** of this angle if

$$C_{\alpha\beta}(r, f) + C_{\alpha\beta}(r, 1/f) = O(1).$$

We note that this condition is equivalent to

$$\sum_{\alpha \le \varphi_k \le \beta} \frac{\sin \frac{\pi}{\gamma}(\varphi_k - \alpha)}{r_k^{\pi/\gamma}} + \sum_{\alpha \le \psi_k \le \beta} \frac{\sin \frac{\pi}{\gamma}(\psi_k - \alpha)}{\rho_k^{\pi/\gamma}} < \infty,$$

where $r_k e^{i\varphi_k}$ are the zeros of $f(z)$, and $\rho_k e^{i\psi_k}$ are its poles.

Definition 2. We shall say that a function $f(z)$, meromorphic for $|z| < \infty$, belongs to the class $\mathfrak{A}_{\alpha\beta}$, if it satisfies the conditions:
1) the zeros and poles of $f(z)$ lying inside the angle $\alpha \le \arg z \le \beta$ are very close to the sides of this angle; 2) there is a representation

$$f(z) = \exp(Q(z))P(z),$$

where $Q(z)$ is an entire function for which

$$\int_1^\infty (\ln^+ T(t, Q))\, t^{-\frac{\pi}{\gamma}-1}\, dt < \infty \qquad (\gamma = \beta - \alpha), \tag{5}$$

$$\int_1^\infty \left(\ln^+ |Q(te^{i\alpha})| + \ln^+ |Q(te^{i\beta})|\right) t^{-\frac{\pi}{\gamma}-1}\, dt < \infty, \tag{6}$$

$$\ln^+ M_{\alpha\beta}(r, Q) = O(r^{\pi/\gamma}) \quad \left(M_{\alpha\beta}(r, Q) = \max_{\alpha < \varphi < \beta} |Q(re^{i\varphi})|\right), \tag{7}$$

and $P(z)$ is a meromorphic function for which

$$\int_1^\infty \left(\ln^+ T(t,P)\right) t^{-\frac{\pi}{\gamma}-1}\,dt < \infty. \tag{8}$$

Theorem 2. The derivative of a function of the class $\mathfrak{A}_{\alpha\beta}$ belongs to the class $\mathfrak{A}_{\alpha\beta}$.

Proof. We first verify that $f'(z)$ satisfies the second condition for membership in $\mathfrak{A}_{\alpha\beta}$. Putting $P_1(z) = Q'(z)P(z) + P'(z)$, we shall have

$$f'(z) = \exp(Q(z))P_1(z),$$

therefore it suffices for us to show that

$$\int_1^\infty \left(\ln^+ T(t,P_1)\right) t^{-\frac{\pi}{\gamma}-1}\,dt < \infty. \tag{9}$$

We have

$$\begin{aligned} T(t,P_1) &\ll T(t,Q') + T(t,P) + T(t,P') + O(1) \ll \\ &\ll T(t,Q) + 2T(t,P) + m(t,Q'/Q) + m(t,P'/P) + O(1). \end{aligned} \tag{10}$$

For the quantities $m(t,Q'/Q)$ and $m(t,P'/P)$, from Nevanlinna' s "lemma on the logarithmic derivative" ([2], p. 61) one obtains the estimates

$$m(t,Q'/Q) \ll O\left(\ln^+[tT(2t,Q)]\right), \qquad m(t,P'/P) = O\left(\ln^+[tT(2t,P)]\right).$$

Relation (9) follows trivially from inequality (10), these estimates, and conditions (5) and (8).

We now verify that the first condition for membership in the class $\mathfrak{A}_{\alpha\beta}$ is also satisfied for $f'(z)$. Since the function $f(z)$ belongs to $\mathfrak{A}_{\alpha\beta}$, we have

$$\bar{C}_{\alpha\beta}(r,f) + C_{\alpha\beta}(r,1/f) = O(1).$$

We need to show that

$$C_{\alpha\beta}(r,f') + C_{\alpha\beta}(r,1/f') = O(1).$$

$C_{\alpha\beta}(r,f') = O(1)$, for obviously $C_{\alpha\beta}(r,f') \ll 2C_{\alpha\beta}(r,f)$. In order to establish that also $C_{\alpha\beta}(r,1/f') = O(1)$, it suffices, in view of the obvious inequality

$$C_{\alpha\beta}(r, 1/f') \ll C_{\alpha\beta}(r, f/f') + C_{\alpha\beta}(r, 1/f),$$

to show that $C_{\alpha\beta}(r, f/f') = O(1)$. Using Theorem N, we obtain the relation

$$C_{\alpha\beta}(r, f/f') \ll S_{\alpha\beta}(r, f/f') = S_{\alpha\beta}(r, f'/f) + O(1) =$$
$$= A_{\alpha\beta}(r, f'/f) + B_{\alpha\beta}(r, f'/f) + C_{\alpha\beta}(r, f'/f) + O(1).$$

In this relation $C_{\alpha\beta}(r, f'/f) = O(1)$, since

$$C_{\alpha\beta}(r, f'/f) \leq C_{\alpha\beta}(r, f) + C_{\alpha\beta}(r, 1/f).$$

Let us verify that $A_{\alpha\beta}(r, f'/f) + B_{\alpha\beta}(r, f'/f) = O(1)$. We have

$$A_{\alpha\beta}(r, f'/f) + B_{\alpha\beta}(r, f'/f) = A_{\alpha\beta}(r, Q' + P'/P) + B_{\alpha\beta}(r, Q'/Q) \leq$$
$$\leq \{A_{\alpha\beta}(r, Q) + A_{\alpha\beta}(r, Q'/Q) + A_{\alpha\beta}(r, P'/P) + O(1)\} +$$
$$+ \{B_{\alpha\beta}(r, Q) + B_{\alpha\beta}(r, Q'/Q) + B_{\alpha\beta}(r, P'/P) + O(1)\}.$$

The quantities $A_{\alpha\beta}(r, Q'/Q)$, $A_{\alpha\beta}(r, P'/P)$, $B_{\alpha\beta}(r, Q'/Q)$, $B_{\alpha\beta}(r, P'/P)$ are $O(1)$ by conditions (5) and (8) and Theorem A. The relations $A_{\alpha\beta}(r, Q) = O(1)$ and $B_{\alpha\beta}(r, Q) = O(1)$ follow trivially, respectively, from (6) and (7). The theorem is proved.

3°. **Proof of Theorem 1.** Obviously, in order for an entire function to satisfy the conditions of Theorem 1, it is necessary and sufficient that it belong to the intersection of the classes $\mathfrak{A}_{0\pi}$ and $\mathfrak{A}_{\pi,2\pi}$. (Both necessity and sufficiency become quite obvious if one takes into account the well-known relation ([2], p. 24) $T(t, P) \leq \ln^+ M(t, P) \leq 3T(2t, P)$.) Therefore Theorem 1 follows immediately from Theorem 2.

A trivial consequence of Theorem 2 is the following theorem, which represents a generalization of Theorem 1:

Theorem 3. *Let a system of rays be given*

$$\arg z = \alpha_j \qquad (j = 1, 2, \dots, n; \quad 0 \leq \alpha_1 < \alpha_2 < \dots < \alpha_n < 2\pi; \quad \alpha_{n+1} = \alpha_1 + 2\pi).$$

The derivative of a meromorphic function belonging to the intersection

$$\mathfrak{A}_{\alpha_1\alpha_2} \cap \mathfrak{A}_{\alpha_2\alpha_3} \cap \dots \cap \mathfrak{A}_{\alpha_n\alpha_{n+1}}$$

also belongs to this intersection.

Let us note that functions satisfying the condition of this theorem are representable in the form $f(z) = \exp(Q(z))P(z)$, where $Q(z)$ is an entire function of order not exceeding $\pi\gamma^{-1}$ ($\gamma = \min(\alpha_{j+1} - \alpha_j)$), satisfying certain additional known requirements, and $P(z)$ is a meromorphic function for which

$$\int_1^\infty (\ln^+ T(t,P))t^{-\frac{\pi}{\Gamma}-1}\,dt < \infty \qquad (\Gamma = \max(\alpha_{j+1} - \alpha_j)).$$

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Received
11 X 1959

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Note: Figure translations are in progress. See original paper for figures.