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## Abstract

## Full Text

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# ON THE THEORY OF MULTIDIMENSIONAL SINGULAR INTEGRAL EQUATIONS

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The present note contains some additions to the general theory of multidimensional singular equations in the space $L_2(E_m)$, constructed by S. G. Mikhlin ($^{1-3}$). These additions are obtained with the aid of results and methods from the theory of normed commutative rings of I. M. Gelfand ($^{4}$).

1. Under certain restrictions ($^{1-3}$), the singular integral equation over the whole $m$-dimensional space $E_m$ $(m \geq 2)$ has the form

$$(A\varphi)(x) = \sum_{n_1,n_2,\dots,n_{m-1}=-\infty}^{\infty} a_{n_1 n_2 \dots n_{m-1}}(x)(S_1^{n_1} S_2^{n_2} \dots S_{m-1}^{n_{m-1}}\varphi)(x) + (T\varphi)(x) = f(x) \quad (x \in E_m), \tag{1}$$

where $\varphi(x)$ $(\in L_2(E_m))$ is the unknown function, $f(x)$ and $a_{n_1 n_2 \dots n_{m-1}}(x)$ are prescribed functions; the first of them belongs to the space $L_2(E_m)$, while the others are continuous functions on the whole space $E_m$ (including the point at infinity); $T$ is a linear completely continuous operator acting in $L_2(E_m)$; $S_j$ $(j = 1, 2, \dots, m-1)$ are the "simplest" $m$-dimensional singular integral operators. They admit the representation

$$S_j\varphi = F^{-1} \exp[iv_j(x)](F\varphi)(x) \quad (j = 1, 2, \dots, m-1),$$

in which $v_1(x), \frac{1}{2}v_2(x), \dots, \frac{1}{2}v_{m-1}(x)$ $(0 \leq v_j \leq 2\pi,\ j = 1, 2, \dots, m-1)$ denote the angular spherical coordinates of the point $x$ $(\in E_m)$; $F$ is the unitary Fourier transform operator in $L_2(E_m)$; finally, the series in (1) converges in the operator norm.

According to the general theory ($^{1-3}$) of multidimensional singular integral equations, to each operator $A$ of the form (1) (generally speaking, under certain additional restrictions ensuring, for example, convergence of the series $\sum \max |a_{n_1 n_2 \dots n_{m-1}}(x)|$) there is associated a continuous function

$$\mathcal{A}(x,\theta_1,\theta_2,\dots,\theta_{m-1}) = \sum_{n_1,n_2,\dots,n_{m-1}=-\infty}^{\infty} a_{n_1 n_2 \dots n_{m-1}}(x) \exp\left[i\sum_j n_j\theta_j\right] \qquad (0 \le \theta_j \le 2\pi), \tag{2}$$

called the **symbolic function** (**symbol**) of the operator $A$. In ($^{1-3}$) it is proved that if the symbol (2) nowhere vanishes, then the Fredholm theorem holds for equation (1), i.e. the operator $A$ is normally solvable and the equations $A\varphi = 0$, $A^*\psi = 0$ ($\varphi, \psi \in L_2(E_m)$) have equal finite numbers of linearly independent solutions. Analogous results were obtained in ($^{1-3}$) also for systems of equations of the form (1).

**2.** For simplicity we shall henceforth put $m = 2$ ($E_2 = E$). Then the operator (1) takes the form

$$A = \sum_{j=-\infty}^{\infty} a_j(x) S^j + T,$$

where*

$$(S\varphi)(x) = \frac{1}{2\pi}\int_E \frac{e^{i\nu(y)}\varphi(y)}{|x-y|^2}\,dy,$$

and $|x|$ and $\nu(x)$ are the polar coordinates of the point $x \in E$. Formula (2), defining the symbol of the operator $A$, takes the form

$$\mathcal{A}(x,\theta) = \sum_{j=-\infty}^{\infty} a_j(x) e^{ij\theta} \qquad (0 \leqslant \theta \leqslant 2\pi).$$

Denote by $\mathfrak{T}$ the set of all linear completely continuous operators and by $\mathfrak{S}_k$ the set of all operators of the form

$$A = \sum_{|j|\leqslant k_A} a_j(x) S^j + T \qquad (T \in \mathfrak{T}),$$

where $k_A$ is a finite number depending on the operator $A$, and $a_j(x)$ ($j = 0, \pm 1, \dots$) are continuous functions on the whole space $E$, including the point at infinity.

As is known $^{(1)}$, the set $\mathfrak{S}_k$ forms a ring, and $AB - BA \in \mathfrak{T}$ for any pair of operators $A, B \in \mathfrak{S}_k$. Hence, in particular, it follows that the symbol of the product of any pair of operators from $\mathfrak{S}_k$ is equal to the product of their symbols.

**Theorem 1.** If the operator $A \in \mathfrak{S}_k$, then**

$$\max_{x \in E,\ 0 \leqslant \theta \leqslant 2\pi} |\mathcal{A}(x, \theta)| \leqslant \inf_{T \in \mathfrak{T}} \|A + T\|. \tag{3}$$

The proof of this theorem is based on the following lemma.

**Lemma.** Let $x_0 (\in E)$ and $\theta_0 (0 \leqslant \theta_0 \leqslant 2\pi)$ be an arbitrary fixed pair of points. Then for any positive numbers $\varepsilon_1, \varepsilon_2$ and $\rho$ there exists an ort $\varphi \in L_2(E)$ possessing the following properties:

a) $\|S\varphi - e^{i\theta_0}\varphi\| < \varepsilon_1$;

b) $\displaystyle\int_{|x - x_0| > \rho} |\varphi(x)|^2 \, dx < \varepsilon_2$.

**3.** Denote by $\mathfrak{S}$ the closure, in the operator norm, of the ring $\mathfrak{S}_k$. The set $\mathfrak{S}$ forms a complete normed ring, and for any pair of operators $A$ and $B \in \mathfrak{S}$ the difference $AB - BA \in \mathfrak{T}$.

Let $A \in \mathfrak{S}$ and $\|A_n - A\| \to 0$, where $A_n \in \mathfrak{S}_k$. Then, by Theorem 1, the sequence of symbols $\mathcal{A}_n(x, \theta)$ converges uniformly on $E \times [0, 2\pi]$ to some continuous function $\mathcal{A}(x, \theta)$ $(x \in E;\ 0 \leqslant \theta \leqslant 2\pi)$, independent of the choice of the sequence $A_n$ tending to the operator $A$. From Theorem 1 it also follows that

$$\max_{x, \theta} |\mathcal{A}(x, \theta)| \leqslant \inf_{T \in \mathfrak{T}} \|A + T\|. \tag{4}$$

Thus, the symbol $\mathcal{A}(x, \theta)$ is uniquely determined by the operator $A (\in \mathfrak{S})$.

The function $\mathcal{A}(x, \theta)$ is naturally called the **symbol** of the operator $A$. It is easy to see that the symbol of a product of operators from $\mathfrak{S}$ is equal to the product of the corresponding symbols.

Denote by $\mathfrak{R}$ the quotient ring $\mathfrak{S}/\mathfrak{T}$, with the norm defined by

$$\|\hat{A}\| = \inf_{T \in \mathfrak{T}} \|A + T\| \qquad (\hat{A} \in \mathfrak{R}), \tag{5}$$

where $\hat{A}$ denotes the residue class containing the operator $A$.

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* The integral is understood in the sense of the principal value (see $^{(1)}$).

** We note that from this theorem, in particular, it follows that the operator $\sum_{|n| \leqslant k} a_n(x) S^n \in \mathfrak{T}$ if and only if

From the properties of the ring $\mathfrak{S}$ indicated above it follows that $\mathfrak{R}$ forms a commutative normed ring. All operators belonging to one and the same residue class $\hat{A}$ have one and the same symbol; we shall denote it by $\hat{\mathcal{A}}(x, \theta)$ $(x \in E;\ 0 \leq \theta \leq 2\pi)$.

Let $x_0 \in \widetilde{E}$ and $0 \leq \theta_0 \leq 2\pi$; then the formula

$$F(\hat{A}) = \hat{\mathcal{A}}(x_0, \theta_0) \qquad (\hat{A} \in \mathfrak{R}) \tag{6}$$

defines a linear multiplicative functional in $\mathfrak{R}$. In view of (4) and (5), this functional is continuous. It is not difficult to prove that all continuous linear multiplicative functionals of the ring $\mathfrak{R}$ are exhausted by functionals of the form (6). Hence it follows:

**Theorem 2.** The set $M_{x_0,\theta_0}$ of all $\hat{A} \in \mathfrak{R}$ for which $\hat{\mathcal{A}}(x_0, \theta_0) = 0$, where $x_0 (\in E)$ and $\theta_0$ $(0 \leq \theta_0 \leq 2\pi)$ is an arbitrary fixed point of $E \times [0, 2\pi]$, forms a maximal ideal in $\mathfrak{R}$. These ideals exhaust all maximal ideals of the ring $\mathfrak{R}$.

Thus the bicompact $\mathfrak{M}$ of all maximal ideals of the ring $\mathfrak{R}$ is topologically equivalent to the product $\widetilde{E} \times [0, 2\pi]$, and the symbol $\hat{\mathcal{A}}(x, \theta)$ is the function of the element $\hat{A} (\in \mathfrak{R})$ on the bicompact of maximal ideals $\mathfrak{M}$:

$$\hat{A}(M_{x,\theta}) = \hat{\mathcal{A}}(x, \theta) \qquad (x \in \widetilde{E},\ 0 \leq \theta < 2\pi).$$

From the fact that the ring $\mathfrak{R}$ is symmetric and has no radical it follows ((4), supplement IV, theorem 3) that every noninvertible element of $\mathfrak{R}$ is a generalized divisor of zero. With the aid of this property of the ring $\mathfrak{R}$, Theorem 2, and the results of the note ($^5$), one proves:

**Theorem 3.** In order that an operator $A \in \mathfrak{S}$ be normally solvable and that at least one of the equations

$$A\varphi = 0, \qquad A^*\psi = 0 \qquad (\varphi, \psi \in L_2(E))$$

have no more than a finite number of linearly independent solutions, it is necessary and sufficient that the symbol of the operator $A$ vanish nowhere:

$$\mathcal{A}(x, \theta) \neq 0 \qquad (x \in \widetilde{E};\ 0 \leq \theta \leq 2\pi). \tag{7}$$

If condition (7) is satisfied, then the equations $A\varphi = 0$, $A^*\psi = 0$ $(\varphi, \psi \in L_2(E))$ have the same number of linearly independent solutions.

**4.** Denote by $L_2^{(r)}(E)$ the Hilbert space of $r$-dimensional vector-functions $f = \{f_j\}_1^r$, with components $f_j \in L_2(E)$, and with scalar product defined by

$$(f, g) = \sum_{j=1}^{r} (f_j, g_j).$$

Then the system of multidimensional singular integral equations will have the form $\mathbf{A}\varphi = f$ $(\varphi, f \in L_2^{(r)}(E))$, where the operator $\mathbf{A}$ is defined in $L_2^{(r)}(E)$

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[*] By $\widetilde{E}$ is denoted the compact space obtained by adjoining to $E$ a point at infinity.

by the matrix $\|A_{jk}\|_1^r$ with elements from $\mathfrak{S}$. The matrix $\|\mathcal{A}_{jk}(x, \theta)\|_1^r$ plays the role of the symbol of the operator $\mathbf{A}$.

**Theorem 4.** Let $\mathbf{A}$ be a linear operator defined in the space $L_2^{(r)}(E)$ by the matrix $\|A_{jk}\|_1^r$ with elements from $\mathfrak{S}$. In order that the operator $\mathbf{A}$ be normally solvable and that at least one of the equations $\mathbf{A}\varphi = 0$, $\mathbf{A}^*\psi = 0$ $(\varphi, \psi \in L_2^{(r)}(E))$ have no more than a finite number of linearly independent solutions, it is necessary and sufficient that

$$\det \|\mathcal{A}_{jk}(x, \theta)\|_1^r \neq 0 \qquad (x \in \tilde{E};\ 0 \leq \theta \leq 2\pi). \tag{8}$$

If condition (8) is satisfied, then the equations $\mathbf{A}\varphi = 0$, $\mathbf{A}^*\psi = 0$ $(\varphi, \psi \in L_2^{(r)}(E))$ have the same finite number of linearly independent solutions.

The theorems given above generalize to the case of any natural $m > 2$.

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*Note: Figure translations are in progress. See original paper for figures.*

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