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# V. P. Elizarov

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**Abstract**

**Full Text**

**V. P. Elizarov**

## RELATIONS BETWEEN GENERALIZED RINGS OF QUOTIENTS

*(Presented by Academician A. I. Mal'cev on 6 VI 1960)*

Let  $R$  be an associative ring, and let  $S_1$  and  $S_2$  be multiplicatively closed systems of its nonzero elements, with respect to which there exist left generalized rings of quotients  $R_{(S_1)}$  and  $R_{(S_2)}$  <sup>(2)</sup>. It is natural to raise the question of how these rings are related to one another. If  $S_1$  and  $S_2$  are not connected by any relations, then it is hard to expect that any relations will arise between  $R_{(S_1)}$  and  $R_{(S_2)}$ . Therefore we shall consider the case when  $S_1 \supset S_2$ .

Denote by  $I_1$  and  $I_2$  the kernels of the homomorphic mappings of the ring  $R$  into  $R_{(S_1)}$  and  $R_{(S_2)}$ . The ring  $R_{(S_i)}$ ,  $i = 1, 2$ , consists of equivalence classes of pairs of the form  $(r, s)$ , where  $r \in R$ ,  $s \in S_i$ , and two pairs  $(r', s')$  and  $(r'', s'')$  are equivalent if the relations  $\bar{s}r' - \bar{r}s' \in I_i$  and  $\bar{s}s'' - \bar{r}s'' \in I_i$  hold, where  $\bar{r} \in R$ ,  $\bar{s} \in S_i$ . An element of the ring  $R_{(S_i)}$  represented by the pair  $(r, s)$  will be denoted by  $\left[\frac{r}{s}\right]_i$ . In the ring  $R_{(S_2)}$  consider the set of elements represented by pairs of the form  $(s', s'')$ , where  $s' \in S_1$ ,  $s'' \in S_2$ , and denote it by  $S_{1(S_2)}$ .  $S_{1(S_2)}$  is a multiplicatively closed system of nonzero elements of the ring  $R_{(S_2)}$ .

**Theorem 1.** *If  $I_{(S_2)}$  is an arbitrary  $S_{1(S_2)}$ -prime ideal of the ring  $R_{(S_2)}$ , then the set  $I$  of numerators of its elements forms an  $S_1$ -prime ideal of the ring  $R$ .*

**Proof.** Let  $\left[\frac{i}{s}\right]_2 \in I_{(S_2)}$ , i.e.  $i \in I$ . For an arbitrary element  $s_2 \in S_2$ , by the definition of multiplication in the ring  $R_{(S_2)}$ , the relation

$$\left[\frac{s}{s_2}\right]_2 \cdot \left[\frac{i}{s}\right]_2 = \left[\frac{si}{ss_2}\right]_2 = \left[\frac{i}{s_2}\right]_2 \in I_{(S_2)}.$$

Consequently, an arbitrary element of  $S_2$  can serve as the denominator of an element from  $I_{(S_2)}$ , in the sense that changing the denominator does not affect the membership of the element in the ideal  $I_{(S_2)}$ . If  $i_1, i_2 \in I$ , then, by the definition of addition in the ring  $R_{(S_2)}$ , for any element  $s_2 \in S_2$  the relation

$$\left[\frac{i_1}{s_2}\right]_2 - \left[\frac{i_2}{s_2}\right]_2 = \left[\frac{s_2 i_1 - s_2 i_2}{s_2 s_2}\right]_2 = \left[\frac{i_1 - i_2}{s_2}\right]_2 \in I_{(S_2)},$$

holds, i.e.  $i_1 - i_2 \in I$ . By the definition of multiplication in the ring  $R_{(S_2)}$ , for any elements  $r \in R$ ,  $i \in I$ , and  $s \in S_2$  the relation

$$\left[ \frac{i}{s} \right]_2 \cdot \left[ \frac{sr}{s} \right]_2 = \left[ \frac{\bar{r}sr}{\bar{s}s} \right]_2 \in I_{(S_2)},$$

holds, where  $\bar{s} \in S_2$ ,  $\bar{r} \in R$ , and  $\bar{s}i - \bar{r}s \in I_2$ . But  $I_2 \subset I$ , since elements with numerators from  $I_2$  are zero elements in the ring  $R_{(S_2)}$ . Then  $\bar{s}ir - \bar{r}sr \in I$ ,  $\bar{r}sr \in I$ , and  $\bar{s}ir \in I$ . But

$$\left[ \frac{ir}{s_2} \right]_2 = \left[ \frac{\bar{s}ir}{ss_2} \right]_2$$

and  $ir \in I$ . From the fact that  $\left[ \frac{i}{s} \right]_2 \in I_{(S_2)}$ , it follows that  $\left[ \frac{si}{ss} \right]_2 \in I_{(S_2)}$  and  $\left[ \frac{si}{s} \right]_2 \in I_{(S_2)}$ . Then from the relation

$$\left[ \frac{sr}{s} \right]_2 \cdot \left[ \frac{si}{s} \right]_2 = \left[ \frac{\bar{r}si}{ss} \right]_2,$$

where  $\bar{s} \in S_2$ ,  $\bar{r} \in R$ , and  $\bar{s}sr - \bar{r}s \in I_2$ , it follows that  $\bar{r}si \in I$  and  $\bar{s}sr - \bar{r}si \in I_2 \subset I$ . Hence  $\bar{s}sr \in I$  and  $ri \in I$ . Consequently,  $I$  is a two-sided ideal of the ring  $R$ .

From the definition of an  $S_{1(S_2)}$ -prime ideal it follows that  $I_{(S_2)} \cap S_{1(S_2)}$  is empty, and hence  $I \cap S_1$  is empty. Let  $r \in R$ ,  $s_1 \in S_1$ , and  $rs_1 \in I$ . For an arbitrary element  $s_2 \in S_2$  we have the equality

$$\left[ \frac{s_2r}{s_2} \right]_2 \cdot \left[ \frac{s_2s_1}{s_2} \right]_2 = \left[ \frac{\bar{r}s_2s_1}{\bar{s}s_2} \right]_2,$$

where  $\bar{r} \in R$ ,  $\bar{s} \in S_2$ , and  $\bar{s}s_2r - \bar{r}s_2 \in I_2 \subset I$ . Then  $\bar{s}s_2rs_1 - \bar{r}s_2s_1 \in I$ . But  $rs_1 \in I$ , i.e.  $\bar{s}s_2rs_1 \in I$  and  $\bar{r}s_2s_1 \in I$ . Therefore,

$$\left[ \frac{rs_1s_2}{\bar{s}s_2} \right]_2 \in I_{(S_2)} \quad \text{and} \quad \left[ \frac{s_2r}{s_2} \right]_2 \in I_{(S_2)},$$

since  $I_{(S_2)}$  is an  $S_{1(S_2)}$ -prime ideal. Hence we obtain that  $r \in I$ . In a similar way, from the equality

$$\left[ \frac{s_2s_1}{s_2} \right]_2 \cdot \left[ \frac{s_2r}{s_2} \right]_2 = \left[ \frac{\bar{r}s_2r}{s_2\bar{s}} \right]_2,$$

where  $\bar{s} \in S_2$ ,  $\bar{r} \in R$ , and  $\bar{s}s_2s_1 - \bar{r}s_2 \in I_2 \subset I$ , and from the assumption that  $s_1r \in I$ , it follows that  $r \in I$ . To complete the proof it remains to show that, for any elements  $r \in R$  and  $s_1 \in S_1$ , there exist elements  $r' \in R$  and  $s' \in S_1$  such that  $s'r - r's_1 \in I$ . For an arbitrary element  $s_2 \in S_2$ , consider the elements

$$\left[ \frac{s_2 r}{s_2} \right]_2 \in R_{(S_2)} \quad \text{and} \quad \left[ \frac{s_2 s_1}{s_2} \right]_2 \in S_{1(S_2)}.$$

By the definition of an  $S_{1(S_2)}$ -prime ideal, there exist elements

$$\left[ \frac{r'_1}{s'_2} \right]_2 \in R_{(S_2)} \quad \text{and} \quad \left[ \frac{s'_1}{s''_2} \right]_2 \in S_{1(S_2)}$$

such that the relation holds

$$\left[ \frac{s'_1}{s''_2} \right]_2 \cdot \left[ \frac{s_2 r}{s_2} \right]_2 - \left[ \frac{r'_1}{s'_2} \right]_2 \cdot \left[ \frac{s_2 s_1}{s_2} \right]_2 = \left[ \frac{\bar{r} s_2 r}{\bar{s} s'_2} \right]_2 - \left[ \frac{\bar{\bar{r}} s_2 s_1}{\bar{\bar{s}} s'_2} \right]_2 = \left[ \frac{\beta' \bar{r} s_2 r - \alpha' \bar{\bar{r}} s_2 s_1}{\beta' \bar{s} s''_2} \right]_2,$$

where  $\dot{s}, \bar{s}, \beta' \in S_2$ ,  $\bar{r}, \bar{\bar{r}}, \alpha' \in R$ ,

$$\bar{s} s'_1 - \bar{r} s_2 \in I_2, \quad \dot{s} r'_1 - \bar{\bar{r}} s_2 \in I_2, \quad \beta' \dot{s} s''_2 - \alpha' \bar{s} s_2 \in I_2.$$

By the condition,

$$\beta' r s_2 r - \alpha' \bar{\bar{r}} s_2 s_1 \in I.$$

But  $\bar{r} s_2 = \bar{s} s'_1 - i_2$ , where  $i_2 \in I_2$ . Then

$$\beta' \bar{s} s'_1 r - \beta' i_2 r - \alpha' \bar{\bar{r}} s_2 s_1 \in I,$$

or

$$\beta' \bar{s} s'_1 r - \alpha' \bar{\bar{r}} s_2 s_1 \in I.$$

But  $\beta' \bar{s} s'_1 \in S_1$ , and  $\alpha' \bar{\bar{r}} s_2 \in R$ , and these elements may be taken as  $s'$  and  $r'$ . The theorem is proved.

Assume now that, along with the condition  $S_1 \supset S_2$ , the relation  $I_1 \supset I_2$  also holds.

**Lemma 1.** The correspondence  $\alpha$ , established by the formula

$$\alpha \left( \left[ \frac{r}{s} \right]_2 \right) = \left[ \frac{r}{s} \right]_1,$$

is an  $S_{1(S_2)}$ -reducing mapping of the ring  $R_{(S_2)}$  into the ring  $R_{(S_1)}$ . The kernel of this mapping is  $I_{1(S_2)}$ .

**Proof.** Let the pairs  $(r, s)$  and  $(r', s')$ , where  $r, r' \in R$  and  $s, s' \in S_2$ , be representatives of one and the same element in the ring  $R_{(S_2)}$ . Since  $S_1 \supset S_2$  and  $I_1 \supset I_2$ , these pairs are representatives of one and the same element in the ring  $R_{(S_1)}$ . Consequently, the mapping  $\alpha$  is single-valued and does not depend on the choice of the representative of the element

$$\begin{bmatrix} r \\ s \end{bmatrix}_2.$$

By direct verification it is established that the mapping  $\alpha$  is a homomorphism of the ring  $R_{(S_2)}$  into the ring  $R_{(S_1)}$  with kernel  $I_{1(S_2)}$ . Let

$$\begin{bmatrix} s_1 \\ s_2 \end{bmatrix}_2$$

be an arbitrary element of  $S_{1(S_2)}$ . By the definition of the ring  $R_{(S_1)}$ , in it there exists an element

$$\begin{bmatrix} s_2 \\ s_1 \end{bmatrix}_1,$$

which is a two-sided inverse for

$$\begin{bmatrix} s_1 \\ s_2 \end{bmatrix}_1.$$

But

$$\alpha \left( \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}_2 \right) = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}_1,$$

i.e.

$$\left( \alpha \left( \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}_2 \right) \right)^{-1} = \begin{bmatrix} s_2 \\ s_1 \end{bmatrix}_1 \in R_{(S_1)}.$$

Let

$$\begin{bmatrix} r \\ s \end{bmatrix}_1$$

be an arbitrary element of  $R_{(S_1)}$ . For any  $s_2 \in S_2$  the equality holds

$$\begin{bmatrix} s_2 \\ s \end{bmatrix}_1 \cdot \begin{bmatrix} r \\ s_2 \end{bmatrix}_1 = \begin{bmatrix} r \\ s \end{bmatrix}_1.$$

Then

$$\begin{bmatrix} r \\ s \end{bmatrix}_1 = \left( \alpha \left( \begin{bmatrix} s \\ s_2 \end{bmatrix}_2 \right) \right)^{-1} \cdot \alpha \left( \begin{bmatrix} r \\ s_2 \end{bmatrix}_2 \right).$$

The lemma is proved.

**Lemma 2.** If the ring  $R_{(S_2)}$  is mapped into some ring  $R'$  by means of some  $S_{1(S_2)}$ -reducing mapping  $\beta$ , then the mapping  $\psi$ , defined by the formula

$$\psi \left( \begin{bmatrix} r \\ s \end{bmatrix}_1 \right) = \left( \beta \left( \begin{bmatrix} s \\ s_2 \end{bmatrix}_2 \right) \right)^{-1} \cdot \beta \left( \begin{bmatrix} r \\ s_2 \end{bmatrix}_2 \right),$$

where  $r \in R$ ,  $s \in S_1$ ,  $s_2 \in S_2$ , is such a homomorphic mapping of the ring  $R_{(S_1)}$  into the ring  $R'$  that

$$\beta(R_{(S_2)}) = \psi(\alpha(R_{(S_2)})).$$

**Proof.** Let  $s'_2 \in S_2$ . By the definition of multiplication in the ring  $R_{(S_2)}$ , we have  $\beta \left( \begin{bmatrix} s_2 \\ s'_2 \end{bmatrix}_2 \cdot \begin{bmatrix} r \\ s_2 \end{bmatrix}_2 \right) = \beta \left( \begin{bmatrix} s_2 \\ s'_2 \end{bmatrix}_2 \right) \cdot \beta \left( \begin{bmatrix} r \\ s_2 \end{bmatrix}_2 \right) = \beta \left( \begin{bmatrix} s_2 r \\ s_2^0 s'_2 \end{bmatrix}_2 \right) = \beta \left( \begin{bmatrix} r \\ s'_2 \end{bmatrix}_2 \right)$ , where  $s_2^0 \in S_2$ ,  $\bar{s}_2 \in R$ , and  $s_2^0 s_2 - \bar{s}_2 s_2 \in I_2$ . Since  $\beta$  is an  $S_{1(S_2)}$ -reducing homomorphism, it follows that  $\beta \left( \begin{bmatrix} r \\ s_2 \end{bmatrix}_2 \right) = \left( \beta \left( \begin{bmatrix} s_2 \\ s'_2 \end{bmatrix}_2 \right) \right)^{-1} \cdot \beta \left( \begin{bmatrix} r \\ s'_2 \end{bmatrix}_2 \right)$ , and  $\beta \left( \begin{bmatrix} s \\ s_2 \end{bmatrix}_2 \right) = \left( \beta \left( \begin{bmatrix} s_2 \\ s'_2 \end{bmatrix}_2 \right) \right)^{-1} \cdot \beta \left( \begin{bmatrix} s \\ s'_2 \end{bmatrix}_2 \right)$ . Then  $\psi \left( \begin{bmatrix} r \\ s \end{bmatrix}_1 \right) = \left( \beta \left( \begin{bmatrix} s \\ s'_2 \end{bmatrix}_2 \right) \right)^{-1} \cdot \beta \left( \begin{bmatrix} s_2 \\ s'_2 \end{bmatrix}_2 \right) \cdot \left( \beta \left( \begin{bmatrix} s_2 \\ s'_2 \end{bmatrix}_2 \right) \right)^{-1} \cdot \beta \left( \begin{bmatrix} r \\ s'_2 \end{bmatrix}_2 \right) = \left( \beta \left( \begin{bmatrix} s \\ s'_2 \end{bmatrix}_2 \right) \right)^{-1} \cdot \beta \left( \begin{bmatrix} r \\ s'_2 \end{bmatrix}_2 \right)$ . Consequently, the formula does not depend on the choice of the element  $s_2 \in S_2$ .

Suppose that the pair  $(r_1, s_1)$  is equivalent to the pair  $(r, s)$ , i.e., there exist elements  $r' \in R$  and  $s' \in S_1$  such that  $s' s - r' s_1 \in I_1$  and  $s' r - r' r_1 \in I_1$ .

Then for any element  $s_2 \in S_2$  we have  $\beta \left( \left[ \begin{smallmatrix} s_2 s' s \\ s_2 s_2 s_2 \end{smallmatrix} \right]_2 \right) = \beta \left( \left[ \begin{smallmatrix} s_2 r' s_1 \\ s_2 s_2 s_2 \end{smallmatrix} \right]_2 \right)$   
and  $\beta \left( \left[ \begin{smallmatrix} s' s_2 \\ s_2 s_2 \end{smallmatrix} \right]_2 \right) \cdot \beta \left( \left[ \begin{smallmatrix} s \\ s_2 \end{smallmatrix} \right]_2 \right) = \beta \left( \left[ \begin{smallmatrix} r' s_2 \\ s_2 s_2 \end{smallmatrix} \right]_2 \right) \cdot \beta \left( \left[ \begin{smallmatrix} s_1 \\ s_2 \end{smallmatrix} \right]_2 \right)$ . Analogously,  
 $\beta \left( \left[ \begin{smallmatrix} s' s_2 \\ s_2 s_2 \end{smallmatrix} \right]_2 \right) \cdot \beta \left( \left[ \begin{smallmatrix} r \\ s_2 \end{smallmatrix} \right]_2 \right) = \beta \left( \left[ \begin{smallmatrix} r' s_2 \\ s_2 s_2 \end{smallmatrix} \right]_2 \right) \cdot \beta \left( \left[ \begin{smallmatrix} r_1 \\ s_2 \end{smallmatrix} \right]_2 \right)$ . Hence it follows that  $\beta \left( \left[ \begin{smallmatrix} r' s_2 \\ s_2 s_2 \end{smallmatrix} \right]_2 \right) = \beta \left( \left[ \begin{smallmatrix} s' s_2 \\ s_2 s_2 \end{smallmatrix} \right]_2 \right) \cdot \beta \left( \left[ \begin{smallmatrix} s \\ s_2 \end{smallmatrix} \right]_2 \right) \cdot \left( \beta \left( \left[ \begin{smallmatrix} s_1 \\ s_2 \end{smallmatrix} \right]_2 \right) \right)^{-1}$ ,  
i.e.  $\beta \left( \left[ \begin{smallmatrix} r \\ s_2 \end{smallmatrix} \right]_2 \right) = \beta \left( \left[ \begin{smallmatrix} s \\ s_2 \end{smallmatrix} \right]_2 \right) \cdot \left( \beta \left( \left[ \begin{smallmatrix} s_1 \\ s_2 \end{smallmatrix} \right]_2 \right) \right)^{-1} \cdot \beta \left( \left[ \begin{smallmatrix} r_1 \\ s_2 \end{smallmatrix} \right]_2 \right)$ . Finally,  
 $\left( \beta \left( \left[ \begin{smallmatrix} s \\ s_2 \end{smallmatrix} \right]_2 \right) \right)^{-1} \cdot \beta \left( \left[ \begin{smallmatrix} r \\ s_2 \end{smallmatrix} \right]_2 \right) = \left( \beta \left( \left[ \begin{smallmatrix} s_1 \\ s_2 \end{smallmatrix} \right]_2 \right) \right)^{-1} \cdot \beta \left( \left[ \begin{smallmatrix} r_1 \\ s_2 \end{smallmatrix} \right]_2 \right)$ . Consequently, the formula does not depend on the choice of representative of the element  $\left[ \begin{smallmatrix} r \\ s \end{smallmatrix} \right]_1$ . The remainder of the proof of the lemma is carried out by direct verification, since from Lemma 1 it follows that  $I_{1(S_2)}$  is an  $S_{1(S_2)}$ -prime ideal of the ring  $R_{(S_2)}$ , and from Theorem 1 it follows that  $I_{1(S_2)}$  belongs to every  $S_{1(S_2)}$ -prime ideal that serves as the kernel of an  $S_{1(S_2)}$ -reducing homomorphism <sup>(2)</sup>. The lemma is proved.

As an immediate consequence of the above, we obtain

**Theorem 2.** *If  $S_1, S_2$  are multiplicatively closed systems of elements of an associative ring  $R$ , with respect to which the left generalized rings of quotients  $R_{(S_1)}, R_{(S_2)}$  exist, and  $S_1 \supset S_2, I_1 \supset I_2$ , then the ring  $R_{(S_1)}$  is the left generalized ring of quotients of the ring  $R_{(S_2)}$  with respect to the system  $S_{1(S_2)}$ .*

An analogous theorem also holds for right generalized rings of quotients.

If  $R$  is a commutative ring, then the condition  $I_1 \supset I_2$  already follows from the condition  $S_1 \supset S_2$ , and the rings  $R_{(S_1)}$  and  $R_{(S_2)}$  always exist. Consequently, <sup>(1)</sup> holds:

**Theorem 3.** *If  $R$  is a commutative ring;  $R_{(S_1)}, R_{(S_2)}$  are its generalized rings of quotients with respect to multiplicatively closed systems  $S_1, S_2$  of its elements, and  $S_1 \supset S_2$ , then the ring  $R_{(S_1)}$  is the generalized ring of quotients of the ring  $R_{(S_2)}$  with respect to the system  $S_{1(S_2)}$ .*

We note that if  $I_1 = I_2$ , then the ring  $R_{(S_1)}$  contains a subring isomorphic to the ring  $R_{(S_2)}$ .

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## CITED LITERATURE

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<sup>2</sup> V. P. **Elizarov**, *Izv. Akad. Nauk SSSR, ser. matem.*, **24**, no. 2, 153 (1960).

*Note: Figure translations are in progress. See original paper for figures.*

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