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Abstract

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PHYSICS

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FLUCTUATING MICROFIELD AND MULTIPLE COLLISIONS IN A GAS OF CHARGED (OR GRAVITATING) PARTICLES*

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1. Various characteristics of multiple scattering in a gas of charged (or gravitating) particles, usually calculated according to the scheme of successive binary collisions, contain integrals that diverge logarithmically at large impact parameters ρ. The cutoff value $\rho_{\max}$ in early works (for example, ([1])) was taken to be equal to the mean interparticle distance $r_0 \sim n^{-1/3}$; in most works on plasma –to the Debye length D; in works ([2,3])–again to r_0. For an uncompensated charge or for a system of stars one often takes $\rho_{\max} \sim R$ (R is the size of the system).

The choice $\rho_{\max} \sim D$ (or $\rho_{\max} \sim R$) is not self-evident. In the main region $\rho > r_0$, successive collisions overlap strongly in time, i.e., they are multiple. Therefore, in principle, a certain mutual compensation of the actions of many field particles on the test particle is possible, which is equivalent to a cutoff at $\rho_{\max} \ll D$ (or $\rho_{\max} \ll R$). There exist phenomena (the broadening of hydrogen spectral lines) in which such compensation actually occurs, and integrals that diverge in the "binary" scheme converge when the joint action of many particles (i.e., the fluctuating microfield) is taken into account, with the region $\rho \gg r_0$ proving integrally insignificant ([4]).

Thus, the question of the choice of $\rho_{\max}$ is inseparable from the question of the relation between multiple and binary collisions. As Spitzer showed ([5]), for a particle initially at rest, $\langle(\Delta v)^2\rangle$ changes under the action of the microfield of a system of moving particles in the same way as follows from the "binary" calculation ([1]); however, r_0 does not enter the result. Therefore Spitzer cuts off the divergent integral at $\rho_{\max} = D$. He also gave a qualitative explanation of the result, indicating that the influence of collisions with $\rho \gg r_0$ is equivalent to the influence of statistical fluctuations of the charge density.

A cutoff at the Debye length is obtained when the back reaction of the test particle on the ensemble of field particles is successively taken into account, in particular when plasma waves are included (see ([6,7]) and the literature cited there).

However, from the point of view of separating out the purely "collisional" contribution to scattering, the question of the role of the fluctuating microfield in the choice of $\rho_{\max}$ is not exhausted. In recent works by Theimer and collaborators ([2,3]), the treatment ([5]) is disputed, and $\langle(\Delta v_\perp)^2\rangle$ is calculated (according to the binary scheme) for a moving test particle in a gas of resting field particles (cf. ([8])). The microfield is taken into account on the basis of a gas model consisting of cells of radius r_0, inside which the field is Coulomb, and outside—rapidly decreasing according to a law connected with the Holtsmark distribution for small $|E|$. The result, naturally, contains a logarithm cut off (in essence) at $\rho_{\max} \sim r_0$.

Let us also mention an interesting work, the first in the circle of questions under consideration, by Chandrasekhar ([9]) (it is discussed neither in ([5]) nor in ([2])), similar to ([5]) in

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formulation of the problem, and with ([2])—in the character of the result. The influence of the microfield on scattering is described in ([9]), in the "nearest-neighbor" approximation, by means of the Holtsmark distribution and the "mean lifetime" of the field $|\mathbf{E}|$, $T(|\mathbf{E}|)$, determined from the theory of Brownian motion. The quantity $\langle(\Delta v)^2\rangle$, proportional (according to the theory of random walks) to $\langle E^2 T(E)\rangle \Delta t$, contains a logarithm, cut off (automatically) at $\rho_{\max} \sim r_0$.

The methods of allowing for the microfield both in ([2,3]) and in ([9]) are erroneous. The point is that the semi-intuitive models and assumptions used in them ignore the following characteristic (and mutually related) features of Coulomb scattering, for which quadratic combinations of the microfield vector are essential*: a) scattering by density fluctuations (encounters with $\rho \gg r_0$); b) long-time correlations of microfield values (see below).

In view of the above, it is appropriate to carry out a direct calculation of the influence of the microfield for a formulation of the problem that contains ([5,9]) and ([2,8]) as limiting cases.

2. Let us consider the scattering of a test particle in a gas of field particles. Let N field particles be distributed ideally randomly throughout a volume V and have a Maxwellian velocity distribution $\mathbf{v}$; q_0, m_0, and $\mathbf{v}_0$ are the charge, mass, and initial velocity of the test particle; q_1, m_1, and v_1 are the charge, mass, and most probable velocity of a field particle. Let, moreover, $|q_0 q_1| \gg \hbar v_{01}$ (classical scattering) and $m_0 < m_1$.

The square of the transverse increment of the momentum of the test particle during a time Δt is readily transformed to the form

$$(\Delta \mathbf{p}_\perp)^2 = \left(q_0 \int_0^{\Delta t} \mathbf{E}_\perp(t)\, dt \right)^2 = 2q_0^2 \int_0^{\Delta t} d\tau \int_0^{\Delta t - \tau} \mathbf{E}_\perp(t)\mathbf{E}_\perp(t+\tau)\, dt, \quad (1)$$

where $\mathbf{E}_\perp(t)$ is the projection of the instantaneous microfield strength onto the plane normal to $\mathbf{v}_0$. Taking the statistical average (denoted by angle brackets) of (1) and taking into account that the resulting correlation coefficient $\langle \mathbf{E}_\perp(t)\mathbf{E}_\perp(t+\tau)\rangle$ depends only on τ, but not on t (see below), we obtain:

$$\langle (\Delta \mathbf{p}_\perp)^2 \rangle = 2q_0^2 \int_0^{\Delta t} (\Delta t - \tau)\langle \mathbf{E}_\perp(t)\mathbf{E}_\perp(t+\tau)\rangle\, d\tau. \quad (2)$$

The microfield is the sum of the fields of individual field particles, these particles are independent, and the mean field of each of them is zero, so that

$$\langle \mathbf{E}_\perp(t)\mathbf{E}_\perp(t+\tau)\rangle = \left\langle \sum_{i,k=1}^{N} \mathbf{E}_{i\perp}(t)\mathbf{E}_{k\perp}(t+\tau) \right\rangle = N\langle \mathbf{E}_{1\perp}(t)\mathbf{E}_{1\perp}(t+\tau)\rangle \quad (3)$$

where $\mathbf{E}_1(t)$ is the field of one (any) field particle at the position of the test particle. It is already clear from (2), (3) that in an ideal gas of charged (or gravitating) particles the averaged action of the resultant microfield on a test particle reduces to the sum of the independent actions of individual field particles—regardless of whether $\rho < r_0$ or $\rho > r_0$.

The correlation coefficient entering into (3) is conveniently calculated in a reference frame attached to the test particle; it remains inertial for $\Delta t \ll \tau_{90^\circ}$, where τ_{90° (see ([1,8])) is the time of multiple scattering through an angle $\pi/2$ (or, in the case under consideration $m_0 < m_1$, the same as the characteristic time of longitudinal slowing down). In view of the predominance of small scattering angles, the relative trajectories may be regarded as rectilinear, i.e.

$$\mathbf{E}_1(t) \simeq q_1 \left[\mathbf{r}_0 + (\mathbf{v} - \mathbf{v}_0)t\right] |\mathbf{r}_0 + (\mathbf{v} - \mathbf{v}_0)t|^{-3}. \quad (4)$$

* In contrast to the line-broadening phenomenon mentioned above ([4]), where (in the adiabatic approximation) only the modulus of the microfield is essential.

The desired correlation coefficient is equal to

$$\langle \mathbf{E}_{1\perp}(t)\mathbf{E}_{1\perp}(t+\tau)\rangle \equiv K_{1\perp}(\tau) = \iint \mathbf{E}_{1\perp}(t)\mathbf{E}_{1\perp}(t+\tau) f(\mathbf{v})\, d\mathbf{v}\, d\mathbf{r}_0 / V. \quad (5)$$

Substitute (4) and integrate with respect to $\mathbf{r}_0$, introducing the new variable $\mathbf{s} = [\mathbf{r}_0 + (\mathbf{v} - \mathbf{v}_0)t]/|\mathbf{v} - \mathbf{v}_0|\tau$. This gives ($\mathbf{k}_0$ and $\mathbf{k}$ are unit vectors of $\mathbf{v}_0$ and $\mathbf{v} - \mathbf{v}_0$):

$$K_{1\perp}(\tau) = \frac{q_1^2}{\tau V} \int \frac{f(\mathbf{v})\, d\mathbf{v}}{|\mathbf{v} - \mathbf{v}_0|} \left\{ \int \frac{\mathbf{s}(\mathbf{s} + \mathbf{k})}{s^3 |\mathbf{s} + \mathbf{k}|^3}\, d\mathbf{s} - \int \frac{(\mathbf{k}_0 \mathbf{s}) \mathbf{k}_0 (\mathbf{s} + \mathbf{k})}{s^3 |\mathbf{s} + \mathbf{k}|^3}\, d\mathbf{s} \right\}. \quad (6)$$

The first of the integrals over $\mathbf{s}$ ($\equiv C_1$) is evaluated by replacing $(\mathbf{s}+\mathbf{k})|\mathbf{s}+\mathbf{k}|^{-3} = -\nabla|\mathbf{s} + \mathbf{k}|^{-1}$ and integrating by parts with the use of Gauss' s theorem and the identity $\Delta(s^{-1}) = -4\pi\delta(\mathbf{s})$; this gives $C_1 = 4\pi$. The second integral ($\equiv C_2$) is similarly reduced to the form

$$C_2 = \int \operatorname{div}\{|\mathbf{s} + \mathbf{k}|^{-1} \mathbf{k}_0 (\mathbf{k}_0 \nabla s^{-1})\}\, d\mathbf{s} - \int |\mathbf{s} + \mathbf{k}|^{-1} \mathbf{k}_0 \{(\mathbf{k}_0 \nabla) \nabla s^{-1}\}\, d\mathbf{s}. \quad (7)$$

The first term reduces to an integral over the surface of a small sphere enclosing the point $\mathbf{s} = 0$, and is equal to $4\pi/3$. The second term ($\equiv C_2^{II}$) is evaluated in spherical coordinates with polar axis $\mathbf{k}$, replacing $\mathbf{k}_0\mathbf{s}$ by the formula of spherical trigonometry, expanding $|\mathbf{s} + \mathbf{k}|^{-1}$ in Legendre polynomials, etc.; this gives $C_2^{II} = -(2\pi/3)(2\cos^2\alpha - \sin^2\alpha)$, where α is the angle between $\mathbf{k}_0$ and $\mathbf{k}$. Thus, the quantity in braces in (6) is equal to $2\pi(1 + \cos^2\alpha)$. Substituting also $f(\mathbf{v}) = \pi^{-3/2} v_1^{-3} \exp(-v^2/v_1^2)$, after a simple integration we obtain

$$K_{1\perp}(\tau) = (4\pi q_1^2/\tau V v_0)\{\pi^{-1/2}(v_1/v_0)\exp(-v_0^2/v_1^2) + (1 - v_1^2/2v_0^2)\operatorname{erf}(v_0/v_1)\}. \quad (8)$$

Introducing the mean density of field particles $n = N/V$ and the smallest flight time for which the curvature of the trajectory is still small,

$$\tau_{\min} \sim |q_0 q_1| \mu^{-1} v_{01}^{-3} \quad (\mu = m_0 m_1/(m_0 + m_1), \quad v_{01} \equiv \langle|\mathbf{v}_0 - \mathbf{v}|\rangle),$$

we finally find:

$$\langle(\Delta\mathbf{p}_\perp)^2\rangle = 8\pi n(q_0^2 q_1^2/v_0)\{\pi^{-1/2}(v_1/v_0)\exp(-v_0^2/v_1^2) + (1 - v_1^2/2v_0^2)\operatorname{erf}(v_0/v_1)\}\Delta t \ln(\Delta t/\tau_{\min}). \quad (9)$$

In the limiting cases this formula takes the form:

$$v_0 \gg v_1: \quad \langle(\Delta\mathbf{p}_\perp)^2\rangle \simeq 8\pi n(q_0^2 q_1^2/v_0)\Delta t \ln(\Delta t/\tau_{\min}), \quad (9a)$$

$$v_0 \ll v_1: \quad \langle(\Delta\mathbf{p}_\perp)^2\rangle \simeq \frac{32}{3}\sqrt{\pi}\, n(q_0^2 q_1^2/v_1)\Delta t \ln(\Delta t/\tau_{\min}). \tag{9b}$$

Up to the form of the logarithm, these expressions coincide, respectively, with the known formula [8] and with the result [5], multiplied by 2/3 (the obvious difference between $\langle(\Delta\mathbf{p}_\perp)^2\rangle$ and $\langle(\Delta\mathbf{p})^2\rangle$).

3. Exactly the same result (9) is, of course, obtained also in the case when the correlation coefficient of the microfield is computed from the formula

$$\langle\mathbf{E}_\perp(0)\mathbf{E}_\perp(\tau)\rangle = \iint \mathbf{E}_{0\perp}\mathbf{E}_{\tau\perp} W(\mathbf{E}_0, \mathbf{E}_\tau)\, d\mathbf{E}_0\, d\mathbf{E}_\tau, \tag{10}$$

where $W(\mathbf{E}_0, \mathbf{E}_\tau)$ is the statistical distribution of the microfield for two instants of time, introduced by Chandrasekhar [10]:

$$W(\mathbf{E}_0, \mathbf{E}_\tau) = (2\pi)^{-6} \iint \exp[i\vec{\rho}\,\mathbf{E}_0 + i\vec{\sigma}\,\mathbf{E}_\tau - nC(\vec{\rho}, \vec{\sigma}; \tau)]\, d\vec{\rho}\, d\vec{\sigma}, \tag{11}$$

$$C(\vec{\rho}, \vec{\sigma}; \tau) = \iint \left\{1 - \exp\left[iq_1\left(\frac{\vec{\rho}\,\mathbf{r}_0}{r_0^3} + \frac{\vec{\sigma}\,(\mathbf{r}_0 + (\mathbf{v} - \mathbf{v}_0)\tau)}{|\mathbf{r}_0 + (\mathbf{v} - \mathbf{v}_0)\tau|^3}\right)\right]\right\} f(\mathbf{v})\, d\mathbf{v}\, d\mathbf{r}_0. \tag{12}$$

Substitution of (11) into (10) and integration with respect to $\mathbf{E}_0, \mathbf{E}_\tau$ give, obviously, δ-functions and their derivatives; this makes it possible to carry out the integration over $\vec{\rho}, \vec{\sigma}$, after which integrals of the form (6), etc., are obtained. In this method of calculation the determining role of the temporal correlations of the microfield is clearly visible.

4. Thus, using a fairly general example, we have shown, first (in the development of (5)), that allowing for the joint action of many field particles on the test particle does not change the "binary" character of the scattering formulas; i.e., multiple collisions **on average** "imitate" pair collisions. The meaning of this is that the quadratic characteristics of scattering on Poisson density fluctuations ($\sim \sqrt{n}$) are proportional to n.

Second, the result (9) has been obtained which, in contrast to those appearing in the literature, is finite and does not require any artificial cutoff; the role of the maximum length in the logarithm (9) is evidently played by $v_{01}\Delta t$. The logarithmic dependence on Δt has a simple meaning: the contribution to $\langle(\Delta p_\perp)^2\rangle$ is made only by those flybys which have time to occur completely during the time Δt. Consequently, also in the binary calculation, since it sums the quantities $(\Delta p_{i\perp})^2$ for **complete** flybys, the logarithm should be cut off in accordance with the "condition of completeness of the flyby," i.e., at $\rho_{\max} \sim v_{01}\Delta t$. Moreover, the usual binary formula (8)

Fig. 1

Figure 1: Fig. 1

$$\langle(\Delta p_\perp)^2\rangle = q_0^2 n v_0 \Delta t \int_0^{\rho_{\max}} 2\pi\rho\, d\rho \left[\int_{-\infty}^{\infty} E_{1\perp}(\rho,t)\, dt\right]^2 \tag{13}$$

Fig. 1

can in general form be reduced to (2), transforming it analogously to (1) and correctly (from the condition of completeness of the flyby) choosing the limits of integration, the normalization volume V, etc.

The logarithmic dependence on Δt expresses a peculiar nondiffusive character of multiple Coulomb (or Newtonian) scattering. Here, in contrast to the case of short-range forces, there is a long-range correlation of "macroscopic" time intervals, due to the fact that, as Δt increases, increasingly distant flybys are included among the complete flybys (see Fig. 1: flybys with ρ in the shaded region play no role for scattering over intervals $v_0\Delta t$, but do play a role for the interval $3v_0\Delta t$).

As is seen from the derivation, result (9) is formally applicable in the region $\tau_{\min} \ll \Delta t \ll \tau_{90}$; for an ideal gas the latter is very broad. But in practice the region of nondiffusiveness for a plasma is substantially cut off from above: already beginning with Δt of the order of the period of plasma oscillations, it is necessary to take into account the back reaction of the test particle on the field particles, which leads to Debye cutoff and, consequently, to diffusiveness of scattering.* It is possible that the nondiffusiveness of scattering is more significant in stellar clusters, whose radius is comparable with $v_{01}\tau_{90}$—the mean free path for scattering through an angle $\pi/2$.

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* Multiple scattering is nondiffusive for angles $\ll (\ln \mathcal{N}/\mathcal{N})^{1/2}$, where $\mathcal{N} \gg 1$ is the number of particles in the Debye sphere.

Note: Figure translations are in progress. See original paper for figures.