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Abstract

Full Text

MATHEMATICS

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CONDITIONS FOR GLOBAL STABILITY FOR CERTAIN NONLINEAR DIFFERENTIAL EQUATIONS OF AUTOMATIC CONTROL

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Consider the system of “indirect automatic control”

$$\frac{dx}{dt} = Ax + a\varphi(\sigma), \quad \frac{d\sigma}{dt} = (b, x) - \rho\varphi(\sigma). \quad (1)$$

Here and below we denote matrices by capital Latin letters, vectors by small Latin letters, and scalar quantities by Greek letters. Exceptions to this rule will be: t –time, V –a Lyapunov function, n –the order of vectors and matrices; a denotes a column vector, a^* a row vector, so that $a^*b = (b, a)$ is the scalar product, ba^* is a matrix. Matrices, vectors, and numbers are assumed real, and the function $\varphi(\sigma)$ continuous and real-valued. The notation $H > 0$ means that H is a symmetric positive-definite matrix.

Systems of “direct automatic control” ^(1,2), more general than (1), and systems “of Aizerman type” ⁽³⁾ can be reduced, except for certain special cases, to the form (1). The exceptional cases correspond in the space of coefficients to a set of measure zero.

We shall assume, as usual, that $\rho > 0$ and that the function $\varphi(\sigma)$ satisfies the condition

$$0 < \mu_1 \leq \varphi(\sigma)/\sigma \leq \mu_2 < +\infty \quad (\sigma \neq 0).$$

In addition, we shall assume that the eigenvalues of the matrix

$$K = \frac{1}{\rho} \left(A^* + \frac{1}{\rho} ba^* \right) \quad (2)$$

lie in the left half-plane. It can be shown ⁽⁴⁾ that this condition means stability of the linearized system (1), $\varphi(\sigma) = \mu\sigma$, for sufficiently large $\mu > 0$, and that, conversely, from the latter condition it follows that the spectrum of the matrix K lies in the left half-plane or on the imaginary axis. Therefore, speaking

somewhat imprecisely, the condition on the matrix K means that system (1) is stable “with a sufficiently fast-acting servomotor.”

Theorem 1. Take an arbitrary matrix $G_0 > 0$ and define matrices $H_0 > 0$, $M(\sigma)$ and vectors $a_0, m(\sigma)$ by the relations*

$$KH_0 + H_0K^* = -G_0, \quad a_0 = H_0a,$$

$$M(\sigma) = \int_0^\sigma \exp \left[K \int_\tau^\sigma \frac{d\sigma_1}{\varphi(\sigma_1)} \right] d\tau, \quad m(\sigma) = \frac{1}{\rho^2 \varphi(\sigma)} KM(\sigma)a_0.$$

* In these relations the integral in the expression for $M(\sigma)$ converges and $\|m(\sigma)\| \leq \text{const}$ for $-\infty < \sigma < +\infty$. The matrix H_0 , as is known, is determined uniquely from G_0 . We also note that in the important case for applications of a piecewise-linear function $\varphi(\sigma)$, the integrals appearing in Theorem 1 are taken in elementary functions.

Suppose that, for some $\varepsilon_0 > 0$ and all σ , the following holds:

$$1 - (G_0^{-1}m(\sigma), b) - \sqrt{(G_0^{-1}m(\sigma), m(\sigma)) \cdot (G_0^{-1}b, b)} \geq \varepsilon_0 > 0. \quad (3)$$

For the global stability of system (1), it is necessary and sufficient that, for all σ , $-\infty < \sigma < +\infty$, the inequality

$$2 \int_0^\sigma (a, M(\sigma)a_0) d\sigma \geq (H_0^{-1}M(\sigma)a_0, M(\sigma)a_0) \quad (4)$$

be satisfied.

We outline the proof of the theorem. We seek a Lyapunov function in the form

$$V = (Hx, x) + 2(h(\sigma), x) + \psi(\sigma),$$

where $H = H^* = \text{const}$, $h(0) = 0$, $\psi(0) = 0$. The necessary and sufficient conditions for positive definiteness are

$$H > 0, \quad \psi(\sigma) > (H^{-1}h(\sigma), h(\sigma)) \quad (\sigma \neq 0). \quad (5)$$

Computing \dot{V} , we find that \dot{V} has the form

$$\dot{V} = -(x^*G(\sigma)x + 2g(\sigma)^*x + \gamma(\sigma)).$$

We do not write out the explicit expressions for $G(\sigma)$, $g(\sigma)$, $\gamma(\sigma)$. The relations

$$\gamma(\sigma) > 0, \quad \gamma(\sigma)G(\sigma) - g(\sigma)g(\sigma)^* > 0 \quad (\sigma \neq 0) \quad (6)$$

guarantee the positive definiteness of the function $(-\dot{V})$. Setting

$$\gamma(\sigma) = \varphi(\sigma)^2, \quad g(\sigma) = -\frac{\varphi(\sigma)}{2\rho}b, \quad H = \lambda H_0, \quad (7)$$

we find $\psi(\sigma)$, and also

$$h(\sigma) = \frac{\lambda}{\rho}M(\sigma)a_0, \quad G(\sigma) = \lambda[G_0 - m(\sigma)b^* - bm(\sigma)^*].$$

The eigenvalues of the matrix $G_0^{-1/2}G(\sigma)G_0^{-1/2}$ are the number λ (of multiplicity $n - 2$) and

$$\lambda \left[1 - (G_0^{-1}m, b) \pm \sqrt{(G_0^{-1}m, m) \cdot (G_0^{-1}b, b)} \right].$$

Therefore, when inequality (3) is fulfilled, (6) will be satisfied for all sufficiently large $\lambda > 0$. Inequality (4) is equivalent (for sufficiently large $\lambda > 0$) to inequalities (5). As $|x| + |\sigma| \rightarrow +\infty$, we have $V \rightarrow +\infty$; by reference to the theorem of E. A. Barbashin and N. N. Krasovskii⁽⁵⁾, the proof of Theorem 1 is completed.

A certain cumbersomeness of the expression on the left-hand side of inequality (3) is due to the fact that, for $h(\sigma)$, we obtained a differential equation. By changing the exposition, one can obtain an algebraic equation for $h(\sigma)$.

Below we shall assume the existence of $\varphi'(\sigma)$. Instead of (7), we put:

$$\gamma(\sigma) = \varphi(\sigma)^2, \quad g(\sigma) = \rho\varphi(\sigma)h'(\sigma) - \frac{\varphi(\sigma)}{\rho}b.$$

Then

$$h(\sigma) = \frac{\varphi(\sigma)}{\rho}K^{-1} \left(-Ha + \frac{b}{2\rho} \right),$$

and the relations (6) will be satisfied,

if the following inequality has a real solution $H = H^*$:

$$-\rho(KH + HK^*) - \mu_0^2 K^{-1} \left(Ha - \frac{b}{2\rho} \right) \left(Ha - \frac{b}{2\rho} \right)^* K^{*-1} + \frac{1}{\rho} (ba^*H + Hab^*) - \frac{1}{\rho^2} bb^* > 0.$$

Here

$$\mu_0^2 = \max \varphi'(\sigma)^2, \quad (-\infty < \sigma < +\infty). \quad (8)$$

Putting $K^{-1}(Ha - b/2\rho) = -u$, we arrive, as in (6), at the resolving equations

$$-\rho(KU + UK^*) = \mu_0^2 uu^* + \frac{1}{\rho}(Kub^* + buK^*),$$

$$Ua + Ku + \frac{b'}{2\rho} = 0. \quad (9)$$

The conditions (5) will be equivalent to the inequalities

$$U > 0, \quad [\rho + 2(u, a)] \int_0^\sigma \varphi(\tau) d\tau \geq (U^{-1}u, u)\varphi(\sigma)^2. \quad (10)$$

We obtain the following result:

Theorem 2. Suppose that the resolving equations (9), where μ_0^2 is determined by relation (8), have real solutions u , $U = U^*$ for all vectors b' sufficiently close to b .

For stability in the large of system (1), it is necessary and sufficient that the inequalities (10) be satisfied.

To solve the resolving equations (9), one should, from the first equation (9), express the elements of the matrix U in terms of the elements of u , and substitute into the second equation (9), which, in scalar notation, will be a system of n quadratic equations with respect to the elements of the vector u . By successively eliminating the unknowns, this system can be reduced to one algebraic equation of degree 2^n .

In the case when the coefficients of system (1) contain parameters, solving the last equation is difficult. In these cases the following theorem may be useful:

Theorem 3. Take an arbitrary matrix $G_0 > 0$ and a vector c , and define the matrix $H > 0$ and vector-functions $p(\sigma)$, $q(\sigma)$, $r(\sigma)$ by the relations

$$KH + HK^* = -G_0,$$

$$p(\sigma) = Ha + \frac{b}{2\rho} - \varphi'(\sigma)c + Kc,$$

$$q(\sigma) = \frac{1}{\sqrt{2}} \left[Ha - \frac{b}{\rho} - \varphi'(\sigma)c \right],$$

$$r(\sigma) = \frac{1}{\sqrt{2}} \left[Ha + \frac{b}{\rho} - \varphi'(\sigma)c \right].$$

Suppose that for any σ the matrix

$$\begin{pmatrix} \langle p, p \rangle - \rho & \langle p, q \rangle & \langle p, r \rangle \\ \langle q, p \rangle & \langle q, q \rangle - \rho & \langle q, r \rangle \\ \langle r, p \rangle & \langle r, q \rangle & \langle r, r \rangle + \rho \end{pmatrix}, \quad (11)$$

where $\langle x, y \rangle = (G_0^{-1}x, y)$, while not positive definite, has a positive determinant. For stability in the large of system (1), it is necessary and sufficient that, for all $\sigma \neq 0$, the inequality

$$[\rho + 2(a, c)] \int_0^\sigma \varphi(\tau) d\tau > (H^{-1}c, c)\varphi(\sigma)^2. \quad (12)$$

We outline the proof. Setting $\gamma(\sigma) = \varphi(\sigma)^2$, $h(\sigma) = \frac{\varphi(\sigma)}{\rho}c$, we obtain, as above, that relations (6) are equivalent to the inequality

$$\rho G_0 - pp^* - qq^* + rr^* > 0,$$

which is equivalent to the condition formulated above for the matrix (11). Inequality (5) is equivalent to inequality (12).

Setting $c = -\frac{1}{2\rho}K^{-1}b$, we arrive at a cruder, but more convenient condition for verification:

Theorem 4. *Let $G_0 > 0$, $H > 0$ be defined as in Theorem 3. Suppose that for $-\infty < \sigma < +\infty$ the following holds*

$$\rho^3 + \langle b, b \rangle > \langle z, z \rangle \rho^4 / 4 > \langle b, b \rangle \langle z, z \rangle - 4 \langle b, z \rangle^2.$$

where

$$z = \rho Ha + \left(\frac{\varphi'(\sigma)}{2} K^{-1} - I \right) b, \quad \langle x, y \rangle = (G_0^{-1}x, y).$$

For stability in the large of system (1), it is necessary and sufficient that for all $\sigma \neq 0$ the inequality

$$4\rho^2 [\rho^2 - (a, K^{-1}b)] \int_0^\sigma \varphi(\tau) d\tau > (H^{-1}K^{-1}b, K^{-1}b)\varphi(\sigma)^2. \quad (13)$$

Remark. It follows from the proof that, when the conditions of Theorems 1-4 are satisfied, a Lyapunov function is constructed effectively. In particular, when, for some σ_0 , the corresponding inequality (3), (10), (12), or (13) is violated, one effectively determines a vector x_0 such that the solution of system (1) with initial conditions $\sigma(0) = \sigma_0$, $x(0) = x_0$ does not tend to the origin as $t \rightarrow +\infty$.

The conditions of each of Theorems 1-4 are satisfied if the feedback coefficient ρ is sufficiently large.

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Note: Figure translations are in progress. See original paper for figures.

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