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Abstract

Full Text

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On the Dirichlet Problem for the Equation $\Delta u = u^2$

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Mathematics

The study of spatial transonic motions of a gas leads to the necessity of investigating the following boundary-value problem:

$$\Delta u = u^2 \quad \text{in the domain } G,$$

$$u = \varphi(s) \quad \text{on the boundary } \Gamma \text{ of the domain } G. \tag{1}$$

The study of this boundary-value problem for a bounded domain is also of independent interest. It is known (see, for example, (^{1})) that the equation $\Delta u = u^2$ has no entire solutions, i.e., solutions without singularities at a finite distance.

In the present note we first establish the existence and uniqueness of small solutions. The proof is carried out on the basis of the contraction mapping principle. Then the existence and uniqueness of nonnegative solutions is proved in the case where $\varphi(s)$ is an arbitrary continuous nonnegative function. It is interesting to note another proof of this theorem, given in the work (^{2})*.

Next the boundary-value problem (1) is investigated in the case $\varphi(s) = +\infty$.

Along with problem (1), let us consider the equation:

$$u(P) = -\int_G K(P,Q)u^2(Q)\,dQ + v(P), \tag{2}$$

where G is a bounded domain of n-dimensional Euclidean space; $K(P,Q)$ is the Green' s function of the Dirichlet problem for the domain G; P, Q are points of the domain G; $v(P)$ is a function harmonic in G and assuming on the boundary Γ the prescribed continuous boundary values $\varphi(s)$.

Denote:

$$K_1 = \max_{P\in G}\int_G K(P,Q)\,dQ, \qquad B = \max_{P\in G}|v(P)|.$$

Theorem 1. If $4BK_1 < 1$, then there exists a unique solution of equation (2) satisfying the condition

$$|u(P)| \le \frac{1-\sqrt{1-4BK_1}}{2K_1}. \tag{3}$$

For the proof, we consider the following method of successive approximations:

$$u_k(P) = -\int_G K(P,Q)u_{k-1}^2(Q)\,dQ + v(P), \qquad u_0(P) = v(P).$$

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* Work (2) became known to us after the completion of the present work.

It is easy to show that

$$|u_k(P)| \leqslant \frac{1-\sqrt{1-4BK_1}}{2K_1},$$

$$\max_{P\in G}|u_k(P) - u_{k-1}(P)| \leqslant (1-\sqrt{1-4BK_1})\max_{P\in G}|u_{k-1}(P) - u_{k-2}(P)|$$

for every $k = 1, 2, \ldots$.

Hence follows the uniform convergence of the sequence of functions $u_k(P)$ to a function $u(P)$, which is the desired solution of equation (2). The solution satisfying condition (3) is unique. The limiting function $u(P)$ is also a solution of problem (1).

If the boundary values $\varphi(s)$ are nonnegative, then for a bounded domain of n-dimensional Euclidean space the following theorem holds.

Theorem 2. *Whatever the nonnegative continuous function $\varphi(s)$ may be, the equation $\Delta u = u^2$ has in the domain G a nonnegative solution which assumes the values $\varphi(s)$ on the boundary Γ. In the class of nonnegative functions this solution is unique.*

Proof. As the method of successive approximations, consider Newton's method according to the scheme

$$\Delta\omega_k - 2u_{k-1}\omega_k = u_{k-1}^2 - \Delta u_{k-1}, \qquad \omega_k\big|_\Gamma = 0,$$

$$u_k(P) = u_{k-1}(P) + \omega(P) \qquad (k = 1, 2, \ldots),$$

where $u_0(P)$ is a function harmonic in the domain G, assuming on the boundary Γ the prescribed nonnegative values $\varphi(s)$.

For $k = 1$ we have:

$$\Delta\omega_1 - 2u_0\omega_1 = u_0^2, \qquad \omega_1\big|_\Gamma = 0.$$

Since $u_0(P) \geqslant 0$, this problem admits a solution ω_1, and, by the maximum principle, $\omega_1(P) \leqslant 0$. Let us show that

$$u_1(P) = u_0(P) + \omega_1(P) \geqslant 0.$$

Indeed, $u_1(P)$ satisfies the conditions:

$$\Delta u_1 = u_0^2 + 2u_0\omega_1 = u_0(u_1 + \omega_1), \qquad u_1\big|_\Gamma = \varphi(s) \geqslant 0.$$

Then, if $u_1 < 0$ at some point of the domain G, there will be a subdomain $G' \subset G$ where $u_1 < 0$ and on whose boundary $u_1 = 0$. In that case, for the domain G' we shall have:

$$\Delta u_1 = u_0(u_1 + \omega_1) \leqslant 0, \qquad u_1 < 0,$$

which is impossible.

Let us note that the equation for $\omega(P)$ can be rewritten in the form

$$\Delta\omega_k - 2u_{k-1}\omega_k = \omega_{k-1}^2, \qquad \omega_k\big|_\Gamma = 0.$$

Therefore, using induction and analogous reasoning, we prove that $u_k(P) \geqslant 0$ and $\omega_k(P) \leqslant 0$ for every k. As a result of this process we obtain a monotone bounded sequence of functions $u(P)$:

$$u_1(P) \geqslant u_2(P) \geqslant \cdots \geqslant u_k(P), \qquad u_k(P) \geqslant 0 \qquad (k = 1, 2, \ldots).$$

At the same time $\omega_k \to 0$, and hence also $u_k^2 - \Delta u_k \to 0$.

Let $u(P)$ be the limiting function of this sequence. Then by standard arguments it is easy to establish that the function $u(P)$ solves the problem

(1). The uniqueness of the solution in the class of nonnegative functions follows from the maximum principle.

Remark. To estimate the difference $u_k(P) - u(P)$, it is enough to find the difference $u_k(P) - v_k(P)$, where $v_k(P)$ is the solution of the problem

$$\Delta v_k(P) = u_k^2(P), \qquad v_k|_\Gamma = \varphi(s).$$

Then

$$u_k(P) - u(P) \leqslant u_k(P) - v_k(P).$$

In the theory of near-sonic motions of a gas, the boundary-value problem (1) with $\varphi(s) = +\infty$ is of special interest.

Theorem 3. *For a bounded domain G of n-dimensional Euclidean space, a solution of problem (1) with $\varphi(s) = +\infty$ exists.*

The proof is based on the following two lemmas.

Lemma 1. *Suppose that on the boundary of the domain G two continuous positive functions $\varphi_1(s)$, $\varphi_2(s)$ are given, with $\varphi_2(s) \geqslant \varphi_1(s)$. Then the nonnegative solution $u_2(P)$, which assumes on the boundary Γ the values $\varphi_2(s)$, is everywhere in G not less than the nonnegative solution $u_1(P)$ corresponding to the boundary values $\varphi_1(s)$.*

The validity of this assertion follows from the maximum principle, since the difference $u_2 - u_1$ satisfies the equation

$$\Delta(u_2 - u_1) - (u_2 + u_1)(u_2 - u_1) = 0,$$

and on the boundary Γ

$$u_2 - u_1 = \varphi_2(s) - \varphi_1(s) \geqslant 0.$$

Lemma 2. *A nonnegative solution $u_N(P)$ of equation (1), assuming on Γ the value N $(N > 0)$, satisfies the inequality*

$$u_N(P) \leqslant v(R),$$

where

$$v(R) = \frac{C(n)}{R^2}, \qquad C(n) = \begin{cases} 24, & n \leqslant 6, \\ 4n, & n > 6; \end{cases}$$

$R = R(P, \Gamma)$ is the distance from the point P to the boundary Γ; P is a point of the domain G.

Proof. Suppose that at some point $P_0 \in G$, $u_N(P_0) > v(P_0)$, where $R_0 = R(P_0, \Gamma)$. Introduce a local coordinate system with origin at the point P_0. Let $(x_1, x_2, \dots, x_n)$ be the coordinates of a variable point Q in this coordinate system.

Consider the function

$$F(Q;R_0)=\frac{C(n)}{R_0^2}\,\frac{1}{\left(\dfrac{x_1^2+\cdots+x_n^2}{R_0^2}-1\right)^2}.$$

For $Q=P_0$ we have $F(P_0;R_0)=v(R_0)$, and as $x_1^2+\cdots+x_n^2\to R_0^2$, $F(Q;R_0)\to+\infty$. In this case there exists a subdomain G^*, contained in the sphere of radius R_0 with center at the point P_0, such that $F(Q;R_0)<u_N(Q)$ for $Q\in G^*$, and on the boundary of the domain G^*, $F=u_N$. But in the domain G^*, $F(Q;R_0)$, as a function of the point Q, satisfies the inequality

$$\Delta F\leqslant F^2.$$

Hence in the domain G^* we have:

$$\Delta(F-u_N)-(F+u_N)(F-u_N)\leqslant 0,$$

and on the boundary of the domain G^*, $F-u_N=0$. Consequently, everywhere in the domain G^* the inequality $F\geqslant u_N$ must hold, which leads to a contradiction.

Proof of Theorem 3. Construct the sequence of functions $u_N(P)$ as $N\to+\infty$. This sequence, by Lemma 1, is increasing and, by Lemma 2, is bounded above by the function $v(R)$. Considering an arbitrary subdomain of G, we see that the limiting function is a solution of the equation $\Delta u=u^2$. It is easy to see that $u(P)\to+\infty$ as the point P approaches the boundary Γ, and moreover $u(P)\leqslant v(R)$.

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