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Abstract

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ON POLYNOMIALS ORTHOGONAL ON AN ARC OF A CIRCLE

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1. We shall consider the asymptotic properties of the polynomials

$$\varphi_n(z) = a_{nn}z^n + \ldots + a_{n0} \qquad (a_{nn} > 0;\ n = 0, 1, 2, \ldots),$$

orthogonal on a given arc l ($\alpha \leqslant \theta \leqslant 2\pi - \alpha$; $\alpha > 0$) of the unit circle $z = e^{i\theta}$. We denote by $w(\theta)$ the weight function ($\geqslant 0$) in the orthogonality relations:

$$\frac{1}{2\pi}\int_{\alpha}^{2\pi-\alpha} \varphi_m(e^{i\theta})\varphi_n(e^{i\theta})w(\theta)\,d\theta = \delta_{mn};$$

together with it we introduce the function

$$t(\theta) = \frac{1}{\sin\frac{1}{2}\theta}\,w(\theta)\sqrt{\cos^2\tfrac{1}{2}\alpha - \cos^2\tfrac{1}{2}\theta}.$$

One may say that we are considering orthogonal polynomials on the whole circle, but assume that the distribution function $\sigma(\theta)$ is absolutely continuous, while $\sigma'(\theta)$ is equal to zero on an entire arc. From such an approach, however, one cannot expect substantial results for our purposes, since in deriving the limiting properties of orthogonal polynomials on the circle it is assumed that

$$\int_0^{2\pi} \ln\sigma'(\theta)\,d\theta > -\infty.$$

Of the known results, here perhaps only one theorem of Szegő concerning Hermitian forms associated with a given curve is applicable. With the aid of this theorem one can prove that

$$\lim_{n\to\infty}\frac{1}{\gamma^n a_{nn}} = \sqrt{1+\sin\tfrac{1}{2}\alpha}\,\exp\left\{\frac{1}{4\pi}\int_{\alpha}^{2\pi-\alpha}\ln t(\theta)\,\frac{\sin\frac{1}{2}\theta\,d\theta}{\sqrt{\cos^2\frac{1}{2}\alpha-\cos^2\frac{1}{2}\theta}}\right\},$$

where $\gamma = \cos\frac{1}{2}\alpha$ is the transfinite diameter of the arc l, while, apart from what has been stated, only membership in L is assumed with respect to $w(\theta)$. To proceed further, it is necessary to find a special class of weight functions for each of which all orthogonal polynomials, beginning with some one, have a sufficiently surveyable form, and then to pass to the limit from these weight functions to "arbitrary" ones. It is by this route that the results of the present note were obtained.

2. Cut the z-plane along the curve l, and map the resulting domain D conformally onto the unit disk of the v-plane by means of the formula

$$z = \frac{(v-\beta)(\beta v - 1)}{(v+\beta)(\beta v + 1)} \qquad \left(\beta = i \operatorname{tg}\frac{\pi - \alpha}{4}\right).$$

The right bank l^+ of the cut l, if one goes from the point $\theta = \alpha$ to the point $\theta = 2\pi - \alpha$, passes into the lower half of the circle $|v| = 1$, and the left bank l^- into the upper one. The argument θ of a point z lying on l, and the argument ω of the corresponding point v on the unit circle, are related by

$$\cos\omega = \frac{\operatorname{tg}\frac{1}{2}\alpha}{\operatorname{tg}\frac{1}{2}\theta}, \qquad d\omega = \frac{\sin\frac{1}{2}\alpha\, d\theta}{2\sin\frac{1}{2}\theta\sqrt{\cos^2\frac{1}{2}\alpha - \cos^2\frac{1}{2}\theta}}. \tag{1}$$

Denote by $g(z) = g(z;t)$ an analytic function, regular and different from zero in the domain D, and such that $g(\infty) > 0$ and $|g(e^{i\theta})|^2 = 1/t(\theta)$ $(\alpha \leqslant \theta \leqslant 2\pi - \alpha)$, where $t(\theta)$ is a prescribed continuous positive function. The function $g(z;t)$ is determined uniquely and, taking account of relations (1), is represented in the form

$$g(z) = g(z;t) = \exp\left\{\frac{1}{2\pi}\int_0^\pi \frac{1 - v^2}{1 - 2v\cos\omega + v^2}\ln\frac{1}{t(\theta)}\,d\omega\right\} \times$$
$$\times \exp\left\{\frac{i}{4\pi}\int_\alpha^{2\pi-\alpha}\frac{\cos\frac{1}{2}\vartheta}{\sqrt{\cos^2\frac{1}{2}\alpha - \cos^2\frac{1}{2}\vartheta}}\ln\frac{1}{t(\theta)}\,d\vartheta\right\}.$$

3. In order to introduce the above-mentioned special weights and the corresponding orthogonal polynomials, put

$$\psi_n(z) = C_n\left\{\frac{v\Omega(v)}{v-\beta}\left(i\,\frac{v-\beta}{1+\beta v}\right)^n + \frac{\Omega(1/v)}{1-\beta v}\left(i\,\frac{1-\beta v}{v+\beta}\right)^n\right\}, \tag{2}$$

where C_n is a constant; $\Omega(v)$ is a rational function of v, real on the real axis. If the function $\Omega(v)$ is such that the only pole in the disk $|v| \leqslant 1$ of the right-hand side of (2) is the point $v = -\beta$, then $\psi_n(z)$ is a polynomial in z. We shall assume, in addition, that all zeros of the function $\Omega(v)$ lie in the domain $|v| < 1$, and all poles in the disk $|v| \leqslant 1$. In this case $\psi_n(z)$ will be exactly of degree n, provided only that $n \geqslant n_\Omega$, where n_Ω depends only on the function $\Omega(v)$. In particular, for $n > n_\Omega$ the leading coefficient of the polynomial $\psi_n(z)$ is equal to

$$b_{nn} = \lim_{v \to -\beta} \frac{\psi_n(z)}{z^n} = \frac{1 + \sin\frac{1}{2}\alpha}{2\sin\frac{1}{2}\alpha} \, \frac{1}{\gamma^n} \, C_n \Omega\left(-\frac{1}{\beta}\right).$$

By means of contour integration it is not difficult to prove that the polynomials $\psi_n(z)$ satisfy, for $n \geqslant n_\Omega$, the orthogonality relations

$$\frac{1}{2\pi} \int_\alpha^{2\pi-\alpha} \psi_n(e^{i\theta}) e^{-im\theta} w_0(\theta)\, d\theta = 0 \qquad (m = 0, 1, 2, \dots, n-1),$$

where the weight function $w_0(\theta)$ is determined by the formula

$$w_0(\theta)\, d\theta = \frac{dv}{iv\Omega(v)\Omega(1/v)} \qquad (v = e^{i\omega}).$$

For $n > n_\Omega$ the normalization condition gives

$$C_n = \sqrt{\frac{2\sin\frac{1}{2}\alpha}{1 + \sin\frac{1}{2}\alpha}} \sqrt{\frac{\Omega(1/\beta)}{\Omega(-1/\beta)}}, \qquad \gamma^n b_{nn} = \sqrt{\frac{1 + \sin\frac{1}{2}\alpha}{2\sin\frac{1}{2}\alpha}} \left|\Omega\left(\frac{1}{\beta}\right)\right|.$$

Introduce, from the function $w_0(\theta)$, the function $t_0(\theta)$, and then the analytic function $g(z; t_0)$. We then find that at any point z of the domain D

$$C_n \frac{\Omega(1/v)}{1 - \beta v} = \frac{1}{2}\sqrt{1 + \sin\frac{1}{2}\alpha}\,(1 + \beta v)\, g(z; t_0).$$

Therefore, for a fixed function $\Omega(\nu)$ and $n \to \infty$, we shall have the asymptotic formula

$$\psi_n(z) \sim \frac{\sqrt{(z+1)^2 - 4\gamma^2 z} + z - 1 - 2\sin\frac{1}{2}\alpha}{2\sqrt{1 + \sin\frac{1}{2}\alpha}\,(z-1)} \left\{\frac{z + 1 + \sqrt{(z+1)^2 - 4\gamma^2 z}}{2\gamma}\right\}^n g(z; t_0), \tag{3}$$

valid in any closed domain lying inside D. In this formula the branch of the radical is taken which is positive for $z = 1$. To study the polynomials at the points of the arc l, we introduce the limiting values $g_+(e^{i\theta};t_0)$, $g_-(e^{i\theta};t_0)$ of the function $g(z;t_0)$ at the point $z=e^{i\theta}$ on l^+, respectively l^-. In addition, put

$$\cos\lambda = \frac{\cos\frac{1}{2}\theta}{\cos\frac{1}{2}\alpha} \quad (0\le\lambda\le\pi).$$

In this notation one can represent $\psi_n(e^{i\theta})$ $(n>n_\Omega)$ in the form

$$\psi_n(e^{i\theta}) = \frac{\sqrt{1-\sin\frac{1}{2}\alpha}\,e^{i\lambda} - \sqrt{1+\sin\frac{1}{2}\alpha}\,e^{-i\cdot\frac{1}{2}\theta}}{2i\sin\frac{1}{2}\theta}\,e^{in(\frac{1}{2}\theta+\lambda)}g_+(e^{i\theta};t_0)+$$

$$+\frac{\sqrt{1-\sin\frac{1}{2}\alpha}\,e^{-i\lambda} - \sqrt{1+\sin\frac{1}{2}\alpha}\,e^{-i\cdot\frac{1}{2}\theta}}{2i\sin\frac{1}{2}\theta}\,e^{in(\frac{1}{2}\theta-\lambda)}g_-(e^{i\theta};t_0). \tag{4}$$

4. Formulas (3) and (4) make it possible to write asymptotic expressions for $\varphi_n(z)$ for an "arbitrary" weight function $\omega(\theta)$. We shall restrict ourselves to conditions of the type of S. N. Bernstein$^{(2)}$. Then our statements read:

Theorem 1. If the function $t(\theta)$ $(\alpha\le\theta\le 2\pi-\alpha)$ is continuous and positive, then, as $n\to\infty$,

$$\varphi_n(z) \sim \frac{\sqrt{(z+1)^2-4\gamma^2 z}+z-1-2\sin\frac{1}{2}\alpha}{2\sqrt{1+\sin\frac{1}{2}\alpha}\,(z-1)}\left\{\frac{z+1+\sqrt{(z+1)^2-4\gamma^2 z}}{2\gamma}\right\}^n g(z;t) \tag{5}$$

uniformly in every closed domain lying entirely inside D.

Theorem 2. If the function $t(\theta)$ $(\alpha\le\theta\le 2\pi-\alpha)$ is positive and for some $\delta>0$ and $k>0$

$$|t(\theta+h)-t(\theta)|\cdot|\ln h|^{1+\delta}<k \qquad (\alpha\le\theta<\theta+h\le 2\pi-\alpha),$$

then, as $n\to\infty$,

$$\varphi_n(e^{i\theta}) \sim \frac{\sqrt{1-\sin\frac{1}{2}\alpha}\,e^{i\lambda} - \sqrt{1+\sin\frac{1}{2}\alpha}\,e^{-i\cdot\frac{1}{2}\theta}}{2i\sin\frac{1}{2}\theta}\,e^{in(\frac{1}{2}\theta+\lambda)}g_+(e^{i\theta};t)+$$

$$+\frac{\sqrt{1-\sin\frac{1}{2}\alpha}\,e^{-i\lambda} - \sqrt{1+\sin\frac{1}{2}\alpha}\,e^{-i\cdot\frac{1}{2}\theta}}{2i\sin\frac{1}{2}\theta}\,e^{in(\frac{1}{2}\theta-\lambda)}g_-(e^{i\theta};t) \tag{6}$$

uniformly on the entire arc l.

It is worth noting that formulas (3), (4), as well as (5), (6), admit passage to the limit $\alpha \to 0$, as a result of which they pass into the known formulas: (3) and (4) pass into the single equality

$$\psi_n(z) = z^n g(z; t_0) \qquad (n > n_\Omega),$$

and (5) and (6) into the single asymptotic relation

$$\varphi_n(z) \sim z^n g(z; t) \qquad (n \to \infty;\ |z| \geq 1).$$

The first step toward proving our theorems consists in introducing a trigonometric sum $s_0(\theta)$ of order $< m$, which on the interval $\alpha \leq \theta \leq 2\pi - \alpha$ gives the best approximation, in the sense of P. L. Chebyshev, to the function

$$\frac{\sin\frac{1}{2}\alpha}{2\sin^2\frac{1}{2}\theta\, t(\theta)}.$$

If we put

$$t_0(\theta) = \frac{\sin\frac{1}{2}\alpha}{2\sin^2\frac{1}{2}\theta\, s_0(\theta)},$$

then, on the basis of known approximation theorems,

$$t(\theta) - t_0(\theta) = O(1)\,\omega(1/m),$$

where $\omega(h)$ is the modulus of continuity of the function $t(\theta)$. The number m is assumed so large that $s_0(\theta) > 0$ everywhere in the interval $\alpha \leq \theta \leq 2\pi - \alpha$. In view of the latter circumstance, the function $s_0(\theta)$ can be represented in the form

$$s_0(\theta) = \Omega(e^{i\omega})\Omega(e^{-i\omega}),$$

where

$$\Omega(v) = A\prod_{k=1}^{2q}(v - c_k)/[(v-\beta)(v+\beta)]^{2q},$$

and all the roots c_k lie in the domain $|v| < 1$, and the complex ones among them are pairwise conjugate; A is a positive constant; $q < m$. With the aid of this function $\Omega(v)$, the polynomial $\psi_m(z)$ is constructed by formula (2) for $n = m$ $(> n_\Omega)$. The second step, and the completion of the proof of both theorems, consist in considering and estimating the difference

$$\frac{1}{a_{mm}}\varphi_m(z) - \frac{1}{b_{mm}}\psi_m(z).$$

Let us note, incidentally, that under the conditions of Theorem 2, by means of the appropriate estimates one proves that

$$\varphi_m(e^{i\theta}) - \frac{a_{mm}}{b_{mm}}\psi_m(e^{i\theta}) = O(1)\left|\ln\frac{1}{m}\right|^{\delta}.$$

5. The simplest particular systems of polynomials $\psi_n(z)$, which are obtained for

$$\Omega(v) = 1, \quad \frac{(v+\beta)(v-\beta)}{v^2-1}, \quad \frac{(v+\beta)(v-\beta)}{v(v-1)},$$

are analogues of the Chebyshev polynomials T_n, U_n, V_n. In particular, the last of these three systems of polynomials, orthogonal on the arc l, was first introduced in an article of the author (³) in connection with a problem in aerodynamics. There some properties of the polynomials were obtained and the corresponding trigonometric moments were given, by means of which the polynomials are represented in the form of determinants. A parametric representation of these polynomials, however, different from the one which we use here, was first given by Ya. L. Geronimus (⁴). The starting point of his investigations is a certain finite-difference equation of the second order. Using our definition (2) of the polynomials $\psi_n(z)$, it is easy to show that, in the most general case, the polynomials $\psi_n(z)$, for $n > n_\Omega$, satisfy the equation

$$\gamma y_{n+2} = (z+1)y_{n+1} - \gamma z y_n.$$

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Note: Figure translations are in progress. See original paper for figures.