THE LIE DERIVATIVE AND THE DIFFERENTIAL EQUATIONS OF THE FIELD OF A GEOMETRIC OBJECT

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Abstract

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THE LIE DERIVATIVE AND THE DIFFERENTIAL EQUATIONS OF THE FIELD OF A GEOMETRIC OBJECT

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A geometric object is a point of any representation space of some group. In the present article we consider a group of analytic transformations of n variables x^i. This group can be defined by a system of n invariant forms ω^i, satisfying the structural equations

$$D\omega^i = [\omega^k \omega^i_k] \qquad (i, j, k = 1, \dots, n).$$

The normal prolongations of the initial representation (x^i) of this group introduce new invariant forms $\omega^i_k, \dots, \omega^i_{k_1 \dots k_p}$, subordinated to the structural equations

$$D\omega^i_{k_1 \dots k_p} = \sum_{\alpha=1}^{p} \frac{1}{\alpha!(p-\alpha)!} [\omega^l_{(k_1 \dots k_\alpha} \omega^i_{k_{\alpha+1} \dots k_p)l}] + [\omega^l \omega^i_{k_1 \dots k_p l}].$$

Any representation $(x^i, \overline{\Gamma}^I)$ is now defined as the space of first integrals of a completely integrable system of forms

$$\omega^i, \qquad \Delta\Gamma^I \equiv d\Gamma^I + \Gamma^{Ik}_i \omega^i_k + \Gamma^{Ikl}_i \omega^i_{kl} + \dots + \Gamma^{Ik_1 \dots k_p}_i \omega^i_{k_1 \dots k_p},$$

where all $\Gamma^{Ik_1 \dots k_\alpha}_i$ are functions of the adjoined invariants Γ^I.

The invariants Γ^I are interpreted as relative, and the transforming $\overline{\Gamma}^I$ as absolute components of the corresponding local geometric object at the point x^i. We shall agree to denote the aggregate of forms $\omega^i_k, \dots, \omega^i_{k_1 \dots k_p}$ briefly by $\omega^i_{K_p}$.

The invariance condition of the geometric object $(x^i, \overline{\Gamma}^I)$ is then written as follows:

$$\omega^i = 0, \qquad d\Gamma^I + \Gamma^{IK_p}_i \omega^i_{K_p} = 0.$$

Let us take as the space of supporting elements some representation space $(x^i, \overline{p}^{\alpha})$ with invariant forms

$$\omega^i, \qquad \Delta p^{\alpha} \equiv dp^{\alpha} + p_i^{\alpha K_q} \omega^i_{K_q}.$$

The space $(x^i, \overline{p}^{\alpha}, \overline{\Gamma}^I)$ is also a representation space of the group. The field of the geometric object $(x^i, \overline{\Gamma}^I)$ in the space of supporting elements $(x^i, \overline{p}^{\alpha})$ is called the manifold $\overline{\Gamma}^I = f^I(x^i, \overline{p}^{\alpha})$ in the space $(x^i, \overline{p}^{\alpha}, \overline{\Gamma}^I)$.

The finite equations of the field immediately lead to the differential equations $\Delta\Gamma^I = \Gamma^I_{\alpha} \Delta p^{\alpha} + \Gamma^I_k \omega^k$, or

$$d\Gamma^I = -\Gamma_i^{IK_p} \omega^i_{K_p} + \Gamma^I_{\alpha} \left(dp^{\alpha} + p_i^{\alpha K_q} \omega^i_{K_q} \right) + \Gamma^I_k \omega^k. \tag{1}$$

By the method of prolongations and envelopes, fields of other geometric objects are invariantly associated with the given field. We have shown that the differential equations (1) and the operations of prolongation and envelope are sufficient for automatically composing Lie derivatives (2) of arbitrary orders of a geometric object of the given field and of any field invariantly associated with it. In this process the Lie derivative is computed in an arbitrary reference system.

Let a vector field $\bar{V}^i = \bar{V}^i(x)$ be given, with respect to which Lie differentiation is carried out. The differential equations of this field, $\Delta V^i \equiv dV^i + V^k \omega^i_k = V^i_k \omega^k$, and their prolongations

$$\Delta V^i_{k_1 \dots k_s} = V^i_{k_1 \dots k_s k} \omega^k \qquad (s = 1, 2, \dots)$$

define a fundamental geometric object $(V^i, V^i_k, \dots, V^i_{k_1 \dots k_s})$ of order s, whose components we shall denote briefly by $(V^i, V^i_{K_s})$.

Theorem 1. *The relative components of the Lie derivative of the field of the geometric object Γ^I are expressed by the right-hand side of equations (1) after replacing in them ω^i by V^i, $\omega^i_{k_1 \dots k_s}$ by $V^i_{k_1 \dots k_s}$, and dp^{α} by zeros:*

$$\overset{L}{D}\Gamma^I = -\Gamma^I{}_i^{K_p} V^i_{K_p} + \Gamma^I_{\alpha p}{}^{\alpha} K_i{}^q V^i_{K_q} + \Gamma^I_k V^k.$$

Corollary 1. *The Lie derivative of the supporting object, as well as of the object $(V^i, V^i_k, \dots, V^i_{k_1 \dots k_s})$, is equal to zero.*

Corollary 2. *If the functions $\Pi^Q = \Phi^Q(\Gamma^I)$ define an envelope of the field of the geometric object Π^Q by the field Γ^I, then*

$$\overset{L}{D}\Pi^Q = \frac{\partial \Phi^Q}{\partial \Gamma^I} \overset{L}{D}\Gamma^I.$$

Corollaries 1 and 2 give

Corollary 3. *The second-order Lie derivative is expressed in terms of the Lie derivative of the prolonged object by the formulas*

$$\overset{L}{D}{}^2\Gamma^I = -\overset{L}{D}(\Gamma^I{}_i^{K_p})V^i_{K_p} + \overset{L}{D}(\Gamma^I_\alpha)^\alpha K_i{}^q V^i_{K_q} + \overset{L}{D}(\Gamma^I_k)V^k.$$

Thus, the prolongation of the operation of Lie differentiation reduces to normal prolongations of equations (1). Knowledge of the finite transformation law of all objects obtained under prolongations and envelopes becomes unnecessary.

The union of the object Γ^I and its Lie derivative $\overset{*}{\Gamma}{}^I = \overset{L}{D}\Gamma^I$ is also a geometric object.

Theorem 2. *Invariant forms of representation* $(x^i, \bar{\Gamma}^I, \bar{\overset{*}{\Gamma}}{}^I)$ *are the forms*

$$\omega^i, \qquad \Delta\Gamma^I = d\Gamma^I + \Gamma^I{}_i^{K_p}\omega^i_{K_p}, \qquad \overset{*}{\Delta}\Gamma^I = d\overset{*}{\Gamma}{}^I + \frac{\partial\Gamma^I{}_i^{K_p}}{\partial\Gamma^K}\overset{*}{\Gamma}{}^K\omega^i_{K_p}.$$

It follows directly from this that the Lie derivative of the object Γ^I forms an independent (and, moreover, linear homogeneous) object only in the case when the object Γ^I is linear.

Example. We shall apply the theory of the Lie derivative given here to finding the invariant form of the Euler equations and of variations of the integral

$$I = \int F\left(x^i, x^i_\alpha, x^i_{\alpha\beta}\right)[dt^1 \dots dt^p] \qquad (i, j, k = 1, \dots, n;\ \alpha, \beta, \gamma = 1, \dots, p),$$

extended over a p-dimensional surface $x^i = \varphi^i(t^\alpha)$ and containing the first and second partial derivatives $x^i_\alpha = \partial x^i/\partial t^\alpha$ and $x^i_{\alpha\beta} = \partial^2 x^i/\partial t^\alpha \partial t^\beta$. The parameter space t^α (as well as the space (x^i)) is subjected to the group of analytic transformations with the corresponding invariant forms θ^α and their normal prolongations $\theta^\alpha_{\beta_1\dots\beta_r}$. The successive prolongations of the differential equations $\omega^i = \lambda^i_\alpha\theta^\alpha$ of the surface give

$$\Delta\lambda^i_\alpha \equiv d\lambda^i_\alpha + \lambda^k_\alpha\omega^i_k - \lambda^i_\beta\theta^\beta_\alpha = \lambda^i_{\alpha\beta}\theta^\beta,$$

$$\Delta\lambda^i_{\alpha\beta} \equiv d\lambda^i_{\alpha\beta} + \lambda^k_{\alpha\beta}\omega^i_k - \lambda^i_{\gamma\beta}\theta^\gamma_\alpha - \lambda^i_{\alpha\gamma}\theta^\gamma_\beta + \lambda^k_\alpha\lambda^l_\beta\omega^i_{kl} - \lambda^i_\gamma\theta^\gamma_{\alpha\beta} = \lambda^i_{\alpha\beta\gamma}\theta^\gamma.$$

The system of forms $(\theta^\alpha, \omega^i, \Delta\lambda^i_\alpha, \Delta\lambda^i_{\alpha\beta})$ is completely integrable and determines the representation space $(t^\alpha, x^i, x^i_\alpha, x^i_{\alpha\beta})$ of the direct product of the groups acting in the spaces (t^α) and (x^i). We write our integral in invariant form:

$$I = \int f[\theta^1 \dots \theta^p].$$

The quantities $(t^\alpha, x^i, x^i_\alpha, x^i_{\alpha\beta}, F)$ together likewise determine the representation space with invariant forms $\theta^\alpha, \omega^i, \Delta\lambda^i_\alpha, \Delta\lambda^i_{\alpha\beta}, df - f\theta^\alpha_\alpha$. The integrand function generates in it a hypersurface with differential equation

$$df - f\theta^\alpha_\alpha = f_\alpha\theta^\alpha + f_i\omega^i + f^\alpha_i\Delta\lambda^i_\alpha + f^{\alpha\beta}_i\Delta\lambda^i_{\alpha\beta}. \tag{2}$$

The first prolongation of equation (2) gives

$$df_\alpha - f_\beta\theta^\beta_\alpha - f_\alpha\theta^\beta_\beta - f\theta^\beta_{\beta\alpha} + f^\beta_k\lambda^\kappa_\gamma\theta^\gamma_{\beta\alpha} + 2f^{\beta\lambda}_k\lambda^\kappa_{\tau\gamma}\theta^\gamma_{\beta\alpha} + f^{\beta\gamma}_k\lambda^\kappa_\tau\theta^\tau_{\alpha\beta\gamma} =$$

$$= f_{\alpha\beta}\theta^\beta + f_{\alpha i}\omega^i + f^\beta_{\alpha i}\Delta\lambda^i_\beta + f^{\beta\gamma}_{\alpha i}\Delta\lambda^i_{\beta\gamma},$$

$$df_i - f_k\omega^k_i - f_i\theta^\alpha_\alpha - f^\alpha_k\lambda^l_\alpha\omega^k_{il} - f^{\alpha\beta}_k\lambda^l_{\alpha\beta}\omega^k_{il} - f^{\alpha\beta}_j\lambda^k_\alpha\lambda^l_\beta\omega^j_{ikl} =$$

$$= f_{\alpha i}\theta^\alpha + f_{ik}\omega^k + f^\alpha_{ik}\Delta\lambda^k_\alpha + f^{\alpha\beta}_{ik}\Delta\lambda^k_{\alpha\beta}, \tag{3}$$

$$df^\alpha_i + f^\beta_i\theta^\alpha_\beta - f^\alpha_k\omega^k_i - f^\alpha_i\theta^\beta_\beta - 2f^{\alpha\beta}_k\lambda^l_\beta\omega^k_{il} + f^{\beta\gamma}_i\theta^\alpha_{\beta\gamma} =$$

$$= f^\alpha_{\beta i}\theta^\beta + f^\alpha_{ki}\omega^k + f^{\alpha\beta}_{ik}\Delta\lambda^k_\beta + f^{\alpha\beta\gamma}_{ik}\Delta\lambda^k_{\beta\gamma},$$

$$df^{\alpha\beta}_i + f^{\gamma\beta}_i\theta^\alpha_\gamma + f^{\alpha\gamma}_i\theta^\beta_\gamma - f^{\alpha\beta}_k\omega^k_i - f^{\alpha\beta}_i\theta^\gamma_\gamma = f^{\alpha\beta}_{\gamma i}\theta^\gamma + f^{\alpha\beta}_{ki}\omega^k + f^{\alpha\beta\gamma}_{ki}\Delta\lambda^k_\gamma + f^{\alpha\beta\tau\sigma}_{ik}\Delta\lambda^k_{\tau\sigma}.$$

At the same time, the conditions for the independence of the integral from the parametric representation must be satisfied:

$$f_\alpha = 0, \qquad f^\alpha_k\lambda^k_\beta + 2f^{\alpha\gamma}_k\lambda^k_{\gamma\beta} - f\delta^\alpha_\beta = 0, \qquad f^{\alpha\beta}_k\lambda^k_\gamma = 0. \tag{4}$$

Now we can form the first and second variations of the integral, taking into account, in accordance with Theorem 1, only equations (2) and (3), the expressions for the forms $\Delta\lambda^i_\alpha, \Delta\lambda^i_{\alpha\beta}$, and setting $\theta^\alpha_{\beta_1\dots\beta_r} = 0$ in the process of forming them:

$$\delta I = \int D^L f[\theta^1 \dots \theta^p] = \int \left(f^i_{iV} + f^\alpha_i \lambda^k_\alpha V^i_k + f^{\alpha\beta}_i (\lambda^k_{\alpha\beta} V^i_k + \lambda^k_\alpha \lambda^l_\beta V^i_{kl}) \right) [\theta^1 \dots \theta^p].$$

Let

$$\lambda^k_\alpha V^i_k \equiv \Lambda^i_\alpha, \qquad \lambda^k_{\alpha\beta} V^i_k + \lambda^k_\alpha \lambda^l_\beta V^i_{kl} \equiv \Lambda^i_{\alpha\beta}.$$

Noting that, by virtue of Corollary 1,

$$D^L \Lambda^i_\alpha = 0 \quad \text{and} \quad D^L \Lambda^i_{\alpha\beta} = 0,$$

we obtain the second variation

$$\delta^2 I = \int D^L_2 f[\theta^1 \dots \theta^p] = \int \left(D^L f_i V^i + D^L f^\alpha_i \Lambda^i_\alpha + D^L f^{\alpha\beta}_i \Lambda^i_{\alpha\beta} \right) [\theta^1 \dots \theta^p],$$

where the Lie derivative of $f_i, f^\alpha_i, f^{\alpha\beta}_i$ is obtained from equations (3):

$$\overset{L}{D} f_i = f_k V^k_i + f^\alpha_k \lambda^l_\alpha V^k_{il} + f^{\alpha\beta}_k \lambda^l_{\alpha\beta} V^k_{il} + f^{\alpha\beta}_j \lambda^k_\alpha \lambda^l_\beta V^j_{ikl} + f_{ik} V^k + f^\alpha_{ik} \Lambda^k_\alpha + f^{\alpha\beta}_{ik} \Lambda^k_{\alpha\beta},$$

$$\overset{L}{D} f^\alpha_i = f^\alpha_k V^k_i + 2 f^{\alpha\beta}_k \lambda^l_\beta V^k_{il} + f^\alpha_{ki} V^k + f^{\alpha\beta}_{ik} \Lambda^k_\beta + f^{\alpha\beta\gamma}_{ik} \Lambda^k_{\beta\gamma},$$

$$\overset{L}{D} f^{\alpha\beta}_i = f^{\alpha\beta}_k V^k_i + f^{\alpha\beta}_{ki} V^k + f^{\gamma\alpha\beta}_{ki} \Lambda^k_\gamma + f^{\alpha\beta\sigma\tau}_{ik} \Lambda^k_{\sigma\tau}.$$

In conclusion, we indicate the invariant form of the Euler equations

$$h_k \equiv f_k - f^\alpha_{k;\alpha} + f^{\alpha\beta}_{k;\alpha;\beta} = 0,$$

where $f^\alpha_{k;\beta} = f^\alpha_{lk} \lambda^l_\beta + f^{\alpha\gamma}_{kl} \lambda^l_{\gamma\beta} + f^{\alpha\sigma\tau}_{kl} \lambda^l_{\tau\sigma\beta}$ is the result of applying to f^α_k the operator of total differentiation with respect to the parameters t^β, and $f^{\alpha\beta}_{k;\tau;\sigma}$ is the result of applying this operator twice to $f^{\alpha\beta}_k$. The quantities h_k form a relative covector whose components, by virtue of (4), are linearly dependent:

$$h_k \lambda^k_\alpha = 0.$$

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Note: Figure translations are in progress. See original paper for figures.