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Abstract

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Hydromechanics

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On the Theory of Anisotropic Turbulence

(Presented by Academician V. A. Fock, 28 March 1960)

The task of a phenomenological theory consists in constructing, in covariant form, a closed system of equations describing experimental facts over as broad a range as possible. In recent years a series of works has appeared on measuring the characteristics of turbulence in the boundary layer. The results of the measurements do not fit within the framework of the simple theory previously proposed by the author ([1]). In the present paper considerations are advanced which generalize the theory and make it capable of describing the new facts.

§ 1. By an isotropic relation between the tensor of turbulent stresses $\Pi_{ik} \equiv -\rho\overline{u_i' u_k'}$, where u_i' are the fluctuating velocities, and the strain-rate tensor $\dot{e}_{ik} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_k} + \frac{\partial \bar{u}_k}{\partial x_i}\right)$, we shall mean a relation of the form

$$\hat{\Pi}_{ik} = 2K\dot{e}_{ik}, \tag{1,1}$$

where $\hat{\Pi}_{ik}$ is the deviator, namely

$$\hat{\Pi}_{ik} \equiv \Pi_{ik} + \Pi\delta_{ik}; \qquad \Pi \equiv -\frac{1}{3}(\Pi_{11} + \Pi_{22} + \Pi_{33}) \tag{1,2}$$

and K is a scalar coefficient of viscosity, which on general grounds should be assumed to be a function of the turbulent pressure Π. The existence of the isotropic relation (1,1) does not mean that the turbulence is isotropic in the generally accepted sense of this term, since the diagonal components of the deviator (1,1), generally speaking, are not equal to zero. The isotropic equations of state (1,1) of the theory ([1]) also describe anisotropic turbulence.

However, in a steady flow in a circular pipe the tensor $\dot{e}_{ik}$ has only one nonzero component, $e_{12} = \frac{1}{2}\frac{du}{dr}$, and therefore (1,1) gives

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$$\hat{\Pi}_{11} = \hat{\Pi}_{22} = \hat{\Pi}_{33} = 0; \qquad \Pi_{12} = K\frac{du}{dy}. \tag{1,3}$$

Laufer ([2]) measured the distributions of $\overline{u'_i u'_k}$ along the radius of a pipe of circular cross-section. It was found that the curves for $\overline{u_1'^2}$, $\overline{u_2'^2}$, and $\overline{u_3'^2}$ do not coincide. They are close in the core of the flow, but diverge appreciably as the wall is approached. All of them have a sharp maximum near the wall. The turbulent pressure Π, calculated from Laufer' s data, increases monotonically from the pipe axis to the wall, reaching a maximum at a distance of the order of $0.01a$ (a is the radius), and then rapidly falls to zero. The maximum value of Π is 4–5 times greater than on the axis. Laufer' s data do not agree with (1,3). He obtained

$$\Pi_{11} \neq \Pi_{22} \neq \Pi_{33}; \qquad \Pi_{12} - K\frac{du}{dy} \equiv F_{12} \neq 0, \tag{1,4}$$

with $F_{12} \cong 0$ in the core of the flow.

Laufer' s data show agreement of the experiment with the isotropic relation (1.1) in the core of the flow and a deviation from it near the wall. We shall call this deviation anisotropic turbulence.

§ 2. Denote the difference of the tensors by

$$\hat{\Pi}_{ik} - 2K\dot{e}_{ik} \equiv \hat{F}_{ik}; \qquad \hat{F}_{ik} = F_{ik} - {}^1\!/_3\, F_{ss}\delta_{ik}. \tag{2.1}$$

F_{ik} is an additional tensor which must describe the observed departure from the isotropic relation; $\hat{F}_{ik}$ is its deviator. For F_{ik} one must formulate equations of state expressing it in terms of the basic quantities, which for an incompressible fluid are

$$\bar{u}_i, \ \Pi, \ \bar{p}. \tag{2.2}$$

In addition, we add two unit vectors: $\mathbf{s}$, in the direction of the flow, and $\mathbf{v}$, along the principal normal to the trajectory; in the boundary layer this second vector may be replaced by $\mathbf{n}$, the normal to the wall directed into the flow.

In forming F_{ik}, we form derivatives from (2.2) and then use multiplication and contraction of tensors; here we assume that the equations of state depend only on the "internal state" of a fluid particle, and therefore we do not use tensors of the form $\bar{u}_i A_k$. In this way a large number of tensors can be constructed. We divide them into two groups: the group of tensors that do not have the character of viscous stresses, i.e., do not depend on $\dot{e}_{ik}$, we shall denote by P_{ik}; they are analogous to thermal stresses in the broad sense of the word (analogous to certain tensors appearing in the known Burnett equations used

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in gas dynamics), and the group of tensors characterizing the viscous stresses proper, i.e., essentially dependent on $\dot{e}_{ik}$, we denote by T_{ik}; they may be given the form

$$T_{ik} = 2K_{ikmn}\dot{e}_{mn}; \qquad \hat{T}_{ik} = 2\hat{K}_{ikmn}\dot{e}_{mn}, \tag{2.3}$$

where K_{ikmn} is a fourth-rank viscosity tensor describing anisotropic viscosity, with

$$\hat{K}_{ikmn} \equiv K_{ikmn} - {}^{1}\!/_{3} K_{ssmn}\delta_{ik}. \tag{2.4}$$

The symmetry of the tensors T_{ik} and $\dot{e}_{mn}$ requires the equalities

$$K_{ikmn} = K_{iknm} = K_{kimn} = K_{kinm}. \tag{2.5}$$

The general form of the deviator of the stress tensor now becomes

$$\hat{\Pi}_{ik} = 2\left(K\delta_{im}\delta_{kn} + \hat{K}_{ikmn}\right)\dot{e}_{mn} + \hat{P}_{ik}. \tag{2.6}$$

We shall narrow the number of tensors to be discussed by subjecting them to the requirement: in the core of the flow the isotropic relation (1.1) is valid and there are no nonviscous stresses, i.e.,

$$\hat{P}_{ik} \to 0; \qquad K_{ikmn} \to 0 \quad \text{in the core of the flow.} \tag{2.7}$$

The simplest tensors P_{ik} satisfying (2.7) are

$$P_{ik}^{(1)} = a_1 \frac{\partial \Pi}{\partial x_i}\frac{\partial \Pi}{\partial x_k}; \qquad P_{ik}^{(2)} = a_2\left(\frac{\partial \bar{p}}{\partial x_i}\frac{\partial \Pi}{\partial x_k} + \frac{\partial \bar{p}}{\partial x_k}\frac{\partial \Pi}{\partial x_i}\right). \tag{2.8}$$

The tensor of anisotropic viscosity, in general, depends on $\dot{e}_{ik}$, Π, $\bar{p}$ and on their derivatives. In forming, by multiplication, the simplest tensors satisfying (2.7), we consider, all other conditions being equal, the terms with second derivatives to be more substantial than those with products of first derivatives, i.e. li-

linear ones to nonlinear ones:

$$K_{ikmn}^{(1)} = a_1\dot{e}_{im}\delta_{kn} + \ldots; \qquad K_{ikmn}^{(2)} = a_2(\dot{e}_{im}\dot{e}_{kn} + \dot{e}_{km}\dot{e}_{in}); \tag{2,9}$$

$$K_{ikmn}^{(3)} = a_3\frac{\partial^2 \dot{e}_{mn}}{\partial x_i \partial x_k}; \qquad K_{ikmn}^{(4)} = a_4\left(\frac{\partial^2 \Pi}{\partial x_i \partial x_m}\dot{e}_{kn} + \ldots\right); \tag{2,10}$$

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we do not write out the terms needed for the symmetrization of (2,5). Using the characteristic directions important in the boundary layer, $\mathbf{s}, \mathbf{n}$, we obtain another viscosity tensor of the form

$$K^{(5)}_{ikmn} = A\,(s_i n_m \dot{e}_{kn} + ...). \tag{2,11}$$

For the tensor of anisotropic viscous stresses, nonlinear dependence on $\dot{e}_{ik}$ is characteristic.

§ 3. The role of the tensors introduced may be estimated by considering plane flow. We place the origin of the axes Oxy on the wall and direct the Ox axis along the flow, whose width is $2l$. Then

$$\overline{\mathbf{u}} = u(y)\mathbf{s}; \qquad \Pi = \Pi(y); \qquad \overline{p} = ax + f(y). \tag{3,1}$$

The constant $a < 0$. The tensors (2,8) take the form

$$\hat{P}^{(1)}_{11} = \hat{P}^{(1)}_{33} = -\frac{1}{2}\hat{P}^{(1)}_{22} = -\frac{a_1}{3}\left(\frac{d\Pi}{dy}\right)^2; \qquad \hat{P}^{(1)}_{12} = 0;$$

$$\hat{P}^{(2)}_{11} = \hat{P}^{(2)}_{33} = -\frac{1}{2}\hat{P}^{(2)}_{22} = -\frac{2}{3}a_2\frac{\partial\overline{p}}{\partial y}\frac{d\Pi}{dy}; \qquad \hat{P}^{(2)}_{12} = a_2\frac{\partial\overline{p}}{\partial x}\frac{d\Pi}{dy}. \tag{3,2}$$

The viscous-stress tensors $T^{(1)}_{ik}, T^{(3)}_{ik}, T^{(4)}_{ik}$ describe only the anisotropy of the diagonal terms; for them $T_{12} = 0$, while the tensors $T^{(2)}_{ik}$ and $T^{(5)}_{ik}$ describe only shear anisotropy; for them the diagonal terms are zero:

$$\hat{T}^{(2)}_{12} = a_2\left(\frac{du}{dy}\right)^3; \tag{3,3}$$

$$\hat{K}^{(5)}_{ik12} = 0, \quad \text{except } \hat{K}^{(5)}_{1212} = A\frac{du}{dy}; \qquad \hat{T}^{(5)}_{12} = A\left(\frac{du}{dy}\right)^2. \tag{3,4}$$

Let us note that Prandtl’ s mixing-length theory, which gives a coefficient of turbulent viscosity of the form

$$\rho l^2\frac{du}{dy}, \tag{3,5}$$

is contained as a special case in the viscosity tensor $\hat{K}^{(5)}_{ikmn}$, as is evident from comparing (3,4) and (3,5). Prandtl’ s theory of viscosity, properly speaking, is a theory of anisotropic viscosity, and its natural generalization to spatial flows is the tensor (2,11), not the invariant J, as Prandtl assumes [3].

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In a plane flow only one quantity—the pressure $\overline{p}$—depends on x (see (3,1)), and a single integration of the averaged equations of motion gives the equation:

$$a(y-l)-\eta\frac{du}{dy}=\hat{P}_{12}+\left[K+A\frac{du}{dy}+a_2\left(\frac{du}{dy}\right)^2\right]\frac{du}{dy}; \qquad (3,6)$$

η is the coefficient of molecular viscosity. To formulate a closed system of equations, the coefficients a_i, α_i, A must be assumed to be functions of the turbulent pressure Π, their form being determined from experiment.

To determine Π as a function of the coordinates and times there is equation (1), which for a plane stationary flow has the form

$$\hat{\Pi}_{12}\frac{du}{dy}-\frac{d}{dy}(I_2+P_2)+\eta\left(\frac{3}{2\rho}\frac{d^2\Pi}{dy^2}-\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}}\right)=0; \qquad (3,7)$$

I_k is the vector of the flux density of turbulent energy

$$I_k\equiv\frac{1}{2}\rho\,\overline{(u_1'^2+u_2'^2+u_3'^2)u_k'}, \qquad (3,8)$$

which is expressed through $\partial\Pi/\partial x_k$. In the work of W. Dryden (4) data are given indicating the existence of an anisotropic relation between I_k and $\partial\Pi/\partial x_i$. This relation in the author' s work (5) is given in the form

$$I_k=-L_{ki}\frac{\partial\Pi}{\partial x_i}; \qquad L_{ki}=L(\Pi)\delta_{ki}+\hat{L}_{ki}, \qquad (3,9)$$

where $\hat{L}_{ki}\to 0$ in the core of the flow. The vector P_k has an analogous structure. Equation (3,7) for a plane flow, using (3,6), takes the form

$$\left[a(y-l)-\eta\frac{du}{dy}\right]\frac{du}{dy}+\frac{d}{dy}\left(L\frac{d\Pi}{dy}\right)+\frac{3}{2}\nu\frac{d^2\Pi}{dy^2}-\eta g\Pi^\alpha=0; \qquad (3,10)$$

g and α are constants of the theory (1). Adding here equation (3,6), we obtain a system of two equations for determining the two functions $u=u(y)$ and $\Pi=\Pi(y)$. The complete coefficient of anisotropic viscosity

$$K+A\frac{du}{dy}+\alpha_2\left(\frac{du}{dy}\right)^2,$$

in which K, A, α_2 are certain functions of Π, satisfies all qualitative requirements. Near the axis of the flow it passes into the invariant K, which is determined by direct measurements in this region; after this, by measurements near the wall one can measure $A+\alpha_2\,du/dy$.

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With regard to the existence in the boundary layer of nonviscous stresses P_{ik}, there is an indication in Dryden' s work (6). Additional experiments are needed here. Their task consists in determining the phenomenological coefficients in the equations of state. If this is done fully (for different Reynolds numbers) from measurements in flumes and pipes, then all other boundary-value problems can be solved by numerical integration of the equations of the theory.

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Note: Figure translations are in progress. See original paper for figures.

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