THE PLANE PROBLEM OF THE THEORY OF ELASTICITY FOR AN INFINITE STRIP WITH STRESSES OR DISPLACEMENTS PRESCRIBED ON THE BOUNDARY

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Abstract

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THEORY OF ELASTICITY

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THE PLANE PROBLEM OF THE THEORY OF ELASTICITY FOR AN INFINITE STRIP WITH STRESSES OR DISPLACEMENTS PRESCRIBED ON THE BOUNDARY

(Presented by Academician S. L. Sobolev, 19 XI 1959)

Let us first consider the problem with displacements prescribed on the boundary of the strip. It is required to find two functions $\varphi(z)$ and $\psi(z)$, regular inside the strip $-\pi/2 < \operatorname{Im} z < \pi/2$ and satisfying the contour condition

$$\varkappa\varphi(z_1) - z_1\overline{\varphi'(z_1)} - \overline{\psi(z_1)} = -f(z_1). \tag{1}$$

Here $z_1 = \xi + i\eta$ is a point of the contour L ($\eta = \pm\pi/2,\ -\infty < \xi < \infty$); $f(z_1)$ is a given function; the constant $\varkappa$ is greater than unity. Mapping the strip onto the right half-plane $\operatorname{Re} s > 0$ by the function $z = \ln s$ and denoting

$$\Phi(s) = \varphi(\ln s), \qquad \Psi(s) = \psi(\ln s) + (\ln s + i\pi)s\Phi'(s), \qquad s = \sigma + i\tau,$$

$$f_1(\tau) + if_2(\tau) = f(\ln i\tau), \qquad A(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{f_1(\tau) + if_2(\tau)}{i\tau - s}\, d\tau, \tag{2}$$

we arrive at condition (3) on the boundary of the right half-plane, which must be satisfied by the functions $\Phi(s)$ and $\Psi(s)$ regular in it:

$$\varkappa\Phi(i\tau) - \delta(\tau)2\pi\tau\overline{\Phi'(i\tau)} - \overline{\Psi(i\tau)} = -f_1(\tau) - if_2(\tau); \tag{3}$$

$$\delta(\tau) = 1 \quad \text{for } \tau > 0; \qquad \delta(\tau) = 0 \quad \text{for } \tau < 0.$$

Proceeding further in the same way as in (^{1}), we compute $\Phi(s)$:

$$\varkappa\Phi(s) = A(s) + \int_0^1 \left[\overline{A\left(\frac{\bar{s}}{t}\right)} + A(st)\right] M(t)\, dt + \int_0^1 \left[A\left(\frac{\bar{s}}{t}\right) + \overline{A(\bar{s}t)}\right] N(t)\, dt; \tag{4}$$

$$M(t) = \frac{1}{\pi\varkappa^2 t}\int_0^\infty \frac{\cos\left(\frac{x}{\pi}\ln t\right)}{\left(\frac{\operatorname{sh} x}{x}\right)^2 - \frac{1}{\varkappa^2}}\,dx, \qquad N(t) = \frac{1}{\pi\varkappa t}\int_0^\infty \frac{\frac{\operatorname{sh} x}{x}\cos\left(\frac{x}{\pi}\ln t\right)}{\left(\frac{\operatorname{sh} x}{x}\right)^2 - \frac{1}{\varkappa^2}}\,dx. \tag{5}$$

For $\varkappa > 1$ this solution is exactly analogous to the solution of the plane problem for a wedge studied in $(^1)$.

Let us now consider the problem with stresses prescribed on the boundary of the strip. In this case $\varkappa = -1$, and the integrals (5) turn out to be divergent.

We shall use the boundary condition for the Goursat functions $\varphi(z)$ and $\psi(z)$ in the form proposed by G. V. Kolosov $(^2)$,

$$\varphi'(z_1) + z_1\overline{\varphi''(z_1)} + \left[\overline{\psi'(z_1)} + \overline{\varphi'(z_1)}\right] = N - iT; \tag{6}$$

N, T are the normal and tangential stresses at the points of the contour of the strip, decreasing at infinity and satisfying the conditions of statics

$$\int_L T\,d\xi = \int_L N\,d\xi = \int_L (N\xi - T\eta)\,d\xi = 0. \tag{7}$$

Putting

$$\Phi_1(s) = \varphi'(\ln s) = \int_0^\infty u(x)e^{-sx}\,dx, \tag{8}$$

$$A_1(s) = \frac{1}{2\pi}\int_{-\infty}^\infty \frac{N - iT}{i\tau - s}\,d\tau = \int_0^\infty F(x)e^{-sx}\,dx,$$

we obtain the integral equation

$$u(x) + \int_0^\infty \frac{y\bar{u}(y)}{(x+y)^2}\,dy = -F(x). \tag{9}$$

This equation is a special case of equation (3)

$$-\varkappa u(x) + \int_0^\infty \frac{y\bar{u}(y)}{(x+y)^2}\,dy = -F(x) \tag{10}$$

for the value of the parameter $\varkappa$ equal to -1. The value $\varkappa = -1$ is a characteristic number for the given integral equation (it lies on the boundary of the continuous spectrum), and because of this difficulties arise in solving the problem.

Considering the integral equation (10) for $\varkappa < -1$, we obtain the solution $\Phi(s)$ in the form (4).

Let us add and subtract in the right-hand side of (4) the expression

$$I(s,\varkappa) = \frac{1}{4\pi^2}\int_0^\infty \left\{A_1\left(\frac{s}{t}\right) - \overline{A_1\left(\frac{\bar{s}}{t}\right)}\right\}\int_{-\infty}^{\infty}\frac{t^{ix/\pi-1}\,dx}{1+\varkappa+1/6\,\varkappa x^2}\,dt =$$

$$= \frac{3}{2\pi\varkappa x_1}\int_0^1 \left\{\left[A_1\left(\frac{s}{t}\right)+A_1(st)\right] - \left[\overline{A_1\left(\frac{\bar{s}}{t}\right)} + \overline{A_1(\bar{s}t)}\right]\right\} t^{x_1/\pi-1}\,dt,$$

where

$$x_1 = \sqrt{6(1+1/\varkappa)}.$$

Let us calculate the limit of $I(s,\varkappa)$ as $\varkappa \to -1$ $(x_1 \to +0)$

$$I(s,\varkappa) = -\frac{3}{2\pi x_1}\int_0^\infty \left[A_1\left(\frac{s}{t}\right) - \overline{A_1\left(\frac{\bar{s}}{t}\right)}\right]\frac{dt}{t} -$$

$$-\frac{3}{2\pi^2}\int_0^1 \left\{\left[A_1\left(\frac{s}{t}\right)+A_1(st)\right] - \left[\overline{A_1\left(\frac{\bar{s}}{t}\right)} + \overline{A_1(\bar{s}t)}\right]\right\}\frac{\ln t}{t}\,dt + O(x_1).$$

The convergence of the last integrals is ensured by the decrease of the prescribed forces N and T at infinity and by the equality to zero of their principal vector. In view of these conditions $A_1(\infty) = A_1(0) = 0$.

We shall prove that the integral with the factor $1/x_1$ is identically equal to zero. By analyticity with respect to the variable s, it is sufficient to carry out this proof for real values of s. For real s

$$A_1\left(\frac{s}{t}\right) - \overline{A_1\left(\frac{s}{t}\right)} = 2i\,\operatorname{Im} A_1\left(\frac{s}{t}\right),$$

$$\operatorname{Im}\int_0^\infty A_1\left(\frac{s}{t}\right)\frac{dt}{t} = \operatorname{Im}\int_0^\infty A_1(\xi)\frac{d\xi}{\xi} = \operatorname{Im}\left[A_1(\xi)\ln\xi\Big|_0^\infty - \int_0^\infty A_1'(\xi)\ln\xi\,d\xi\right] =$$

$$= -\operatorname{Im}\int_0^\infty A_1'(\xi)\ln\xi\,d\xi = -\operatorname{Im}\frac{1}{2\pi}\int_{-\infty}^{\infty}(N-iT)\int_0^\infty\frac{\ln\xi\,d\xi}{(i\tau-\xi)^2}\,d\tau =$$

$$= -\operatorname{Im}\left\{\frac{i}{2\pi}\int_{-\infty}^{\infty}(N-iT)\ln(-i\tau)\frac{d\tau}{\tau}\right\} = \operatorname{Im}\left\{\frac{1}{2\pi i}\int_L (N-iT)\,\bar{z}_1\,dz_1\right\} =$$

$$= -\frac{1}{2\pi}\int_L (N\xi - T\eta)\,d\xi = -\frac{\text{principal moment of the external forces}}{2\pi} = 0.$$

Thus, passing to the limit as $x \to -1$, we obtain the following formula for the function $\Phi(s)$:

$$-\Phi(s) = A_1(s) + \int_0^1 \left[A_1\left(\frac{s}{t}\right) + A_1(st)\right]\left[M_0(t) + \frac{3\ln t}{2\pi^2 t}\right] dt +$$

$$+ \int_0^1 \left[\overline{A_1\left(\frac{\overline{s}}{t}\right)} + \overline{A_1(\overline{s}t)}\right]\left[N_0(t) - \frac{3\ln t}{2\pi^2 t}\right] dt. \qquad (11)$$

In this formula

$$M_0(t) = \frac{1}{\pi^2 t}\int_0^\infty \left[\frac{1}{\left(\frac{\operatorname{sh} x}{x}\right)^2 - 1} - \frac{3}{x^2}\right]\cos\left(\frac{x}{\pi}\ln t\right) dx,$$

$$N_0(t) = -\frac{1}{\pi^2 t}\int_0^\infty \left[\frac{\frac{\operatorname{sh} x}{x}}{\left(\frac{\operatorname{sh} x}{x}\right)^2 - 1} - \frac{3}{x^2}\right]\cos\left(\frac{x}{\pi}\ln t\right) dx. \qquad (12)$$

The functions $M_0(t)$ and $N_0(t)$ are analytic and vanish at $t = 0$ (see Table 1).

Table 1

t	$-M_0$	$-N_0$	t	$-M_0$	$-N_0$	t	$-M_0$	$-N_0$
0,0005	0,023	−0,023	0,025	−0,063	0,063	0,25	−0,246	0,125
0,001	0,033	−0,033	0,05	−0,1204	0,1142	0,3	−0,220	0,108
0,002	0,037	−0,037	0,08	−0,158	0,141	0,4	−0,222	0,076
0,003	0,036	−0,036	0,1	−0,174	0,1480	0,55	−0,218	0,035
0,005	0,026	−0,026	0,125	–	0,1514	0,7	−0,212	0,002
0,007	0,0147	−0,0147	0,15	−0,198	0,149	0,85	−0,204	−0,023
0,01	−0,002	0,002	0,2	−0,210	0,139	1,0	−0,1975	−0,042

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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