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On Free Solvable Groups

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Abstract

Full Text

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Groups isomorphic to the factor group $F/F^{(n)}$ of a free group F by its n -th commutator subgroup $F^{(n)}$ are called free n -step solvable groups. Below some properties of these groups are derived, and on their basis it is further shown that the elementary theories of free n -step solvable noncommutative groups are recursively undecidable in the sense of Tarski ⁽¹⁾.

No. 1. Auslander and Lyndon ⁽²⁾ proved that if the factor group F/A of a free group F by some normal divisor A of it is infinite, then the center of the factor group $F/[A, A]$ is trivial. Using this result, one easily establishes

Theorem 1. *Let A be such a normal divisor of a free group F that F/A has no torsion. Then any commuting elements u, v of the group $F_0 = F/[A, A]$ either are contained in $A_0 = A/[A, A]$, or are powers of one and the same element of F_0 .*

Considering, instead of F , its subgroup generated by u, v, A , we reduce the matter to the case where the factor group F/A is abelian with two generators u, v . Now the following cases are possible: 1) $F/A = 1$; 2) F/A is free abelian with free generators u, v ; 3) F/A is free cyclic. In the first case $u, v \in A_0$, as was required. In the second case, denote by a, b the generators of an auxiliary metabelian group H . Since F is free, there will be a homomorphism α of the group F into H , under which $u^\alpha = a, v^\alpha = b$. Since α maps A into $[H, H]$ (= the center of H), it follows that $[a, b] = [u, v]^\alpha = 1$, which contradicts the noncommutativity of H . Finally, in the third case, denoting by cA the generating element of the group F/A , we shall have in F_0 : $u = c^k a_1, v = c^l a_2, a_1, a_2 \in A$. Let $kl \neq 0, s = k/(k, l), t = l/(k, l)$. Then $u^t = v^s a, a \in A$, where a commutes with u in F_0 . But this means that a is a central element in the group $\{u, A\}/[A, A]$. Since $\{u, A\}$ is free and the index of A in it is infinite, by the cited result of Auslander–Lyndon, $a \in [A, A]$, and consequently $u^t = v^s$ in F_0 . Let $sx + ty = 1, w = u^{xv^y}$; then $u = w^s, v = w^t$, as required. Similar arguments show that the case $kl = 0, u, v \notin A'$ is impossible.

For a free group F all factors $F/F^{(i)}$ have no torsion. Therefore, putting in Theorem 1 $A = F^{(n-1)}$, we obtain, in particular, that *any two commuting elements of a free solvable group either belong to the last nonidentity commutator subgroup of this group or are powers of one and the same element of it.*

In turn, it follows directly from this that *if some element of a free solvable group commutes with all its conjugates, then it belongs to the last nonidentity commutator subgroup of the group.*

According to Kontorovich ⁽³⁾, a group is called an R -group if, for its elements, from $x^m = y^m$ ($m \neq 0$) it follows that $x = y$.

Theorem 2. *If the factor group F/A of a free group F by its normal divisor is an R -group, then $F_0 = F/[A, A]$ is also an R -group.*

Let $x, y \in F$, $x^m \equiv y^m \pmod{A'}$. Then $x^m \equiv y^m \pmod{A}$, whence, by the assumption, we have $x \equiv y \pmod{A}$, i.e. $y = xa$ ($a \in A$), and $(xa)^m \equiv x^m \pmod{A'}$. Since from $(xa)^m = x^m$ it follows that $a^{x^{m-1}} \cdots a^x a = 1$, where $a^x = x^{-1}ax$, from this, by transforming with the aid of x , we obtain $a^{x^m} \cdots a^{x^2} a^x = 1$, and, consequently, $a^{x^m} = a$; hence in F_0 we have $x^m a = ax^m$. This means that a lies in the center of the group $\{x^m, A\}/[A, A]$. The group $\{x^m, A\}$ is free, and $\{x^m, A\}/A'$ is torsion-free. By the above-mentioned theorem of Auslander-Lyndon we have $a \in A'$, i.e. $y = x$ in F_0 .

Corollary. *All free solvable groups are R -groups.*

For one-step solvable (abelian) free groups the assertion is obvious. By induction, suppose that the assertion is true for n -step solvable free groups. Let F be a free group. Then $F^{(n)}$ is a normal divisor of F , the factor group by which, $F/F^{(n)}$, by the induction hypothesis, is an R -group. By Theorem 2 we conclude that the $(n + 1)$ -step solvable free group $F/F^{(n+1)}$ is an R -group.

§ 2. Let K be a class of models of the signature

$$S = \{P_1, \dots, P_s, a_1, \dots, a_t\},$$

where P_1, \dots, P_s are predicate symbols, and a_1, \dots, a_t are individual constant symbols. Suppose that in each K -model \mathfrak{M} a distinguished submodel is singled out in some way, which we shall call a σ -model. We shall agree to say that the σ -submodels are **elementary in the class K** if there exists a formula $\mathfrak{A}(x)$ of the restricted predicate calculus, whose nonlogical symbols are contained in S , having a unique free individual variable x and such that in every K -model \mathfrak{M} , for every $x \in \mathfrak{M}$, the formula $\mathfrak{A}(x)$ is true if and only if $x \in \sigma(\mathfrak{M})$.

When considering classes of groups it is assumed that the signature consists of the symbols of multiplication and inversion. If, however, one speaks of groups or classes of groups **with fixed elements**, this means that the symbols of the indicated fixed elements are included in the signature of the corresponding group or class of groups.

Lemma. *The successive commutants of free solvable groups are their elementary subgroups.*

Indeed, if G is a free n -step solvable group, then, according to Theorem 1, we have

$$x \in G^{(n-1)} \leftrightarrow (\forall y) (x \cdot y^{-1}xy = y^{-1}xy \cdot x),$$

$$x \in G^{(n-2)} \leftrightarrow (\forall y) (xy^{-1}xy \cdot x^{-1}y^{-1}x^{-1}y \in G^{(n-1)}),$$

.....

$$x \in G' \leftrightarrow (\forall y) (xy^{-1}xy \cdot x^{-1}y^{-1}x^{-1}y \in G''). \tag{1}$$

It is interesting that in the general case the commutant ceases to be an elementary subgroup. Namely, the following remark holds.

Remark. *The commutant is not an elementary notion in any axiomatizable class K of groups among whose factor groups there are free metabelian groups with an arbitrarily large finite number of free generators.*

For the proof we introduce the formulas

$$\mathfrak{P}_m(x) \leftrightarrow (\exists y_1 \dots y_m z_1 \dots z_m) (x = [y_1, z_1] \dots [y_m, z_m]) \quad (m = 1, 2, \dots).$$

Denote by \mathfrak{S} the system of closed formulas of the restricted predicate calculus characterizing the class K , and suppose that the notion of commu-

in K is defined by some formula $\mathfrak{P}(x)$. Then the infinite system $\{\mathfrak{C}, \{\overline{\mathfrak{P}}(c)\}, \mathfrak{P}(c)\}$ (c an individual object symbol) will be contradictory, and therefore, by the local theorem, some finite subsystem of the indicated system will be contradictory. This means that in K the implication

$$(\forall x)(\mathfrak{P}(x) \rightarrow \overline{\mathfrak{P}}_s(x))$$

will hold, where $\overline{\mathfrak{P}}_s(c)$ is the formula with the largest index among the formulas $\overline{\mathfrak{P}}_m(c)$ occurring in the mentioned subsystem. In other words, in K -groups all elements of the commutant must be representable as a product of s commutators. Let H be a free metabelian group with $4s$ free generators, and let G be a group of class K of which H is a quotient group. Direct calculations easily show that every element of the commutant H is a product of $4s - 1$ commutators, but that in H there exist elements which cannot be represented as a product of s commutators. Therefore also in G there are elements of the commutant not representable as a product of s commutators. Thus, in G the formula $\mathfrak{P}(x) \rightarrow \overline{\mathfrak{P}}_s(x)$ does not hold, and the assertion is thereby proved.

It is easy to indicate an infinite set of axiomatizable classes of groups in which the notion of commutant is elementary. Such, for example, are the classes K_m , where K_m is characterized by the axiom

$$(\forall x)(\mathfrak{P}_{m+1}(x) \rightarrow \mathfrak{P}_m(x)).$$

In the class K_m the commutant is characterized by the formula $\mathfrak{P}_m(x)$. As was mentioned, all metabelian groups with $m + 1$ generators belong to K_m , while free metabelian groups with a sufficiently large number of free generators do not belong to K_m . Therefore among the classes K_m there are infinitely many distinct ones.

Elementary subgroups of groups of classes whose signature contains no individual object symbols are characteristic subgroups, and finding them seems interesting not only for particular axiomatizable classes, but also for the most important individual groups.

No. 3. Let K be a class of models of signature S . Denote by $\mathcal{E}(S)$ the totality of all closed formulas of the narrow predicate calculus whose signature is contained in S , and by $\mathcal{E}(K)$ the totality of those formulas from $\mathcal{E}(S)$ which are true on every model of the class K . $\mathcal{E}(K)$ is called the elementary theory of the class K . The elementary theory of a class K is called (recursively) decidable if there exists an algorithm which, for every given formula from $\mathcal{E}(S)$, allows one to decide the question of its membership in $\mathcal{E}(K)$.

Theorem 3. *The elementary theory of a free n -step solvable group with fixed free generators a_1, \dots, a_t , for $n \geq 2$, $t \geq 2$, is undecidable.*

The words “fixed free generators” mean that the signature of the group is assumed to consist of the multiplication symbol and individual object symbols a_1, \dots, a_t . For brevity, below, instead of a_1, a_2 we write a, b . We now introduce the predicates

$$Z(x) \leftrightarrow xa = ax,$$

$$T(x, y) \leftrightarrow xa = ax \ \& \ yb = by \ \& \ xy^{-1} \in G',$$

where instead of $xy^{-1} \in G'$ one should understand the corresponding formula (1). On the basis of Theorem 1, in the group G we have

$$Z(x) \leftrightarrow (\exists m)(x = a^m),$$

$$T(x, y) \leftrightarrow (\exists m)(x = a^m \ \& \ y = b^m).$$

Finally, we introduce in G new operations $+$, $*$ by the formulas

$$z = x + y \leftrightarrow z = xy,$$

$$z = x * y \leftrightarrow (\exists uv)(uxb = xbu \ \& \ T(y, v) \ \& \ uv^{-1}z^{-1} \in G'). \quad (2)$$

If we now restrict ourselves to elements of $Z(x)$, i.e., to elements of the form a^m , then we obtain $a^s + a^t = a^{s+t}$, $a^s * a^t = a^{st}$. Indeed, the first follows directly from (2), and from $x = a^s$, $y = a^t$, $z = a^r$, $uxb = xbu$, $T(y, v)$ we obtain $u = (a^s b)^k$, $v = b^t$, which, together with the relation $uv^{-1}z^{-1} \in G'$, gives $k = t$, $r = st$. Thus, the elementary theory of numbers turns out to

be weakly interpretable in the elementary theory of the group G with fixed elements a, b . The elementary theory of the arithmetic of integers, by Church's theorem, is undecidable. Therefore the elementary theory of the group G is also undecidable.

From the weak interpretability of the arithmetic of integers in the elementary theory of the group G , in the usual way we obtain (see (1)) that the elementary theory of G will be undecidable even without fixed a, b , and also that there exist finitely axiomatizable classes of n -step solvable groups for every $n = 2, 3, \dots$ with essentially undecidable theories.

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Note: Figure translations are in progress. See original paper for figures.

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