NORMAL SPACES AS IMAGES OF ZERO-DIMENSIONAL SPACES

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Abstract

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NORMAL SPACES AS IMAGES OF ZERO-DIMENSIONAL SPACES

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In this paper the following is proved.

Main theorem. *Every normal space X of weight τ is the image of a zero-dimensional (in the sense of small inductive dimension* ind*) set $D \subset D^{\tau}$ under some continuous closed, bicompact, irreducible mapping.*

Addendum. *Since $D \subset D^{\tau}$, the weight of the space D does not exceed τ; since X is normal, D is also normal.*

If X is locally bicompact, respectively paracompact, countably paracompact, or strongly paracompact, then, by virtue of known results (see, for example, ([1])), D will automatically possess the same property. Hence, in particular, it follows that in the case of a strongly paracompact X the set D will be zero-dimensional also in the sense of $\dim D = 0$.

1. We pass to the proof of the main theorem. A canonically closed set, as is known, is any set A which is the closure of some open set (for which one may always take the open kernel JA of the set A).

A **canonical covering** of a space is any finite covering whose elements are canonically closed sets with disjoint open kernels.

If a canonical covering α' **follows** a canonical covering α, i.e. is inscribed in it, then: a) each element of the covering α' is contained in a unique element of the covering α, and b) each element of the covering α is the sum of the elements of the covering α' contained in it. In other words, the covering α' is a **subdivision** of the covering α. Every set of canonical coverings of a given space we shall regard as partially ordered only in the sense of the just mentioned order relation. The set $\Sigma = \{\alpha\}$ of canonical coverings of the space X will be called **refining** if the following condition is satisfied: whatever the point $x \in X$ and its neighborhood Ox, there is a covering α such that the star of the point x in the covering α (i.e. the sum of the elements of the covering α containing the point x) is contained in Ox. A directed refining set of canonical coverings of the space X is called simply a **chain of subdivisions of the space X**.

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* By D^τ is denoted the "generalized Cantor discontinuum of weight τ," i.e. a bicompact space which is the topological product of τ spaces D_α, each of which consists of a finite number of isolated points; moreover, it is usually assumed (which does not impair generality) that D_α consists of two isolated points; for us it is more convenient to abandon this restriction.

A mapping $f : D \to X$ is called **bicompact** if the preimage $f^{-1}x$ of every point $x \in X$ is bicompact. A mapping f of the space D onto the space X is called **irreducible** if under the mapping f no closed set $F \subset D$ distinct from D is mapped onto the whole of X.

We shall easily prove the following proposition:

Lemma 1. *In every normal space X of weight τ there exists a chain of decompositions of cardinality τ.*

First of all: into every open cover ω one can inscribe a canonical one (this is proved, for example, in ([2]), Ch. 6, § 4). Let $\mathfrak{B}$ be a base of cardinality τ of the space X. Consider the system Ω of all possible "binary" covers of the form $\omega = \{U, X \setminus [V]\}$, where U and V, $[V] \subseteq U$, are elements of the base $\mathfrak{B}$. The cardinality of the system Ω is equal to τ. Into each $\omega \in \Omega$ we inscribe a canonical α_ω. It is easy to see that the resulting set $\Sigma_\Omega = \{\alpha_\omega\}$ of canonical covers is refining. In order to obtain from it a directed refining set of canonical covers (i.e., the chain of decompositions of cardinality τ which we need), it suffices to augment the system Σ_Ω by all possible finite products of the covers entering into it, understanding by the product $\alpha_1 \cdots \alpha_n$ of canonical covers $\alpha_1, \dots, \alpha_n$ the canonical cover α, whose elements are the closures of all possible nonempty sets of the form $\Gamma_1 \cap \dots \cap \Gamma_n$, where Γ_i is the open kernel of some element of the cover α_i. Lemma 1 is proved.

2. Along with the space X of weight τ we shall consider its bicompact extension $\overline{X}$ of the same weight τ. Let in $\overline{X}$ there be given a chain of decompositions $\overline{\Sigma} = \{\overline{\alpha}\}$, $\overline{\alpha} = \{\overline{A}_1^\alpha, \dots, \overline{A}_{s_\alpha}^\alpha\}$, of cardinality τ. Then $\Sigma = \{\alpha\}$, where $\alpha = \{A_1^\alpha, \dots, A_{s_\alpha}^\alpha\}$, $A_i^\alpha = X \cap \overline{A}_i^\alpha$, is a chain of decompositions of the space X, and moreover $\overline{A}_i^\alpha = [A_i^\alpha]$ and the order $\alpha' > \alpha$ and $\overline{\alpha}' > \overline{\alpha}$ is one and the same.

For each $\alpha = \{A_1^\alpha, \dots, A_{s_\alpha}^\alpha\}$ consider the finite set

$$D_\alpha = \{1^\alpha, 2^\alpha, \dots, s^\alpha\}$$

of natural numbers furnished with the index α, and for $\alpha' > \alpha$ in Σ define the projection $\mathfrak{D}_\alpha^{\alpha'} : D_{\alpha'} \to D_\alpha$, putting $\mathfrak{D}_\alpha^{\alpha'} j^{\alpha'} = i^\alpha$, where $i^\alpha = i$ is the number of the unique element $A_i^\alpha \in \alpha$ containing the given element $A_j^{\alpha'} \in \alpha'$.

Thus we obtain an inverse spectrum $\{D_\alpha, \mathfrak{D}_\alpha^{\alpha'}\}$ with limit space $\overline{D} = \lim(D_\alpha, \mathfrak{D}_\alpha^{\alpha'})$, which is a bicompactum lying in the topological product $D^\tau = \prod_{\alpha \in \Sigma} D_\alpha$. Therefore the weight of the bicompactum $\overline{D}$ does not exceed

τ. Let $\xi = \{i_\alpha\} \in \overline{D}$. The system $\{\overline{A}^\alpha_{i_\alpha}\}$ is a centered system of closed sets of the bicompactum $\overline{X}$, and therefore their intersection is nonempty. Since the system $\overline{\Sigma} = \{\overline{\alpha}\}$ is refining, the intersection $\bigcap_\alpha \overline{A}^\alpha_{i_\alpha}$ consists of only one point $\overline{x} = \bigcap_\alpha \overline{A}^\alpha_{i_\alpha} \in \overline{X}$, and we put $\overline{f}\xi = \overline{x}$. This defines the mapping $\overline{f} : \overline{D} \to \overline{X}$.

Denote by $U^\alpha_i = \mathcal{E}(\xi \in \overline{D},\ i_\alpha = i)$ the totality of those points $\xi \in \overline{D}$ for which $i_\alpha = i$. These $\overline{U}^\alpha_i$, $i = 1, 2, \dots, s^\alpha$, are, for a given α, disjoint bicompacta forming the cover $\tilde{\delta}_\alpha$ of the bicompactum $\overline{D}$. The totality of all $\overline{U}^\alpha_i$ (over all possible α) forms a base of the bicompactum $\overline{D}$. Moreover, obviously,

$$\overline{f}\,\overline{U}^\alpha_i \subseteq \overline{A}^\alpha_i. \tag{1}$$

Hence, and from the fact that the system $\overline{\Sigma} = \{\overline{\alpha}\}$ is refining, it follows immediately that the mapping $\overline{f} : \overline{D} \to \overline{X}$ is continuous.

3. Let us call a point $\xi = \{i_\alpha\} \in \overline{D}$ *marked* if $\bigcap_\alpha A^\alpha_{i_\alpha} \neq \Lambda$; then necessarily

$$\bigcap_\alpha A^\alpha_{i_\alpha} = \bigcap_\alpha \overline{A}^\alpha_{i_\alpha} = \overline{f}\xi \in X.$$

Denote by D the set of all marked points $\xi \in \overline{D}$, and put

$$U^\alpha_i \cap D = D\,\overline{U}^\alpha_i, \qquad \delta_\alpha = \{U^\alpha_1, \dots, U^\alpha_{s_\alpha}\}.$$

We shall prove that for every point $x \in X$ we have $\overset{-1}{\overline{f}}x \subset D$. Indeed, if for some point $\xi = \{i_\alpha\} \in \overline{D}$ we have $\overline{f}\xi = \bigcap_\alpha \overline{A}^\alpha_{i_\alpha} = x \in X$, then

$$x = \bigcap_\alpha (X \cap \overline{A}^\alpha_{i_\alpha}) = \bigcap_\alpha A^\alpha_{i_\alpha},$$

and this means precisely that ξ is a marked point, i.e. $\xi \in D$.

It follows at once that, denoting by $f : D \to X$ the mapping $\overline{f}$ considered only on D, we obtain a continuous **compact** mapping of the space D into X.

We shall prove the equality

$$fU^\alpha_i = A^\alpha_i. \tag{2}$$

This will also prove that f is a mapping of the set D **onto** the whole space X (and, consequently, $\overline{f}$ is a mapping of the space $\overline{D}$ onto the whole $\overline{X}$); moreover, from (2) it follows that all U^α_i are nonempty and, hence, D is everywhere dense in $\overline{D}$.

The inclusion $fU^\alpha_i \subset A^\alpha_i$ follows from (1).

It remains to prove the more difficult assertion: for every point $x \in A_i^{\alpha_0}$ there is a point $\xi \in U_i^{\alpha_0}$ such that $f\xi = x$.

We turn to this proof. Write $\Sigma = \{\alpha\}$ as a well-ordered set whose first element is α_0:

$$\Sigma = \{\alpha_0, \alpha_1, \dots, \alpha_\lambda, \dots\}, \qquad \lambda < \omega(\tau).$$

For brevity write $A(0)$ instead of $A_i^{\alpha_0}$, and $A(\mu)$ instead of $A_{i_\mu}^{\alpha_\mu}$. Suppose that for all $\mu < \lambda$ the $A(\mu) \in \alpha_\mu$ have been chosen so that $x \in A(\mu)$, and that, whatever ordinal numbers $\mu_1, \dots, \mu_r < \lambda$, taken in a finite number, may be, there exists in some α_ν following all $\alpha_{\mu_1}, \dots, \alpha_{\mu_r}$ (i.e. inscribed in them) a set $A(\nu) \ni x$, contained in all $A(\mu_1), \dots, A(\mu_r)$; here it is not at all required that $\nu < \lambda$. We shall prove that among the elements of the covering α_λ containing the point x one can find such an $A(\lambda)$ that, whatever ordinal numbers $\mu_1, \dots, \mu_r$, smaller than $\lambda + 1$, may be, there again exists a covering α_ν following all $\alpha_{\mu_1}, \dots, \alpha_{\mu_r}$, and in it an element $A(\nu) \ni x$, contained in all $A(\mu_1), \dots, A(\mu_r)$. Suppose that no such $A(\nu)$ can be found. This means that for any

$$A_i \in \alpha_\lambda, \qquad i = 1, 2, \dots, s^{\alpha_\lambda},$$

one can find such a finite collection of sets

$$(i) \qquad A(\mu_1^i), \dots, A(\mu_{r(i)}^i), \qquad \mu_1^i, \dots, \mu_{r(i)}^i < \lambda,$$

that there exists no $A(\nu) \ni x$ contained in all the sets (i) and, in addition, in A_i. The union of all the systems (i), $i = 1, 2, \dots, s^{\alpha_\lambda}$, is a finite system σ of sets $A(\mu)$, $\mu = \mu_1, \dots, \mu_r < \lambda$; consequently, for it there exists an $\alpha_{\nu'}$ following all $\alpha_{\mu_1}, \dots, \alpha_{\mu_r}$, and in it some $A(\nu') \ni x$, contained in all $A(\mu) \in \sigma$. Take a covering α_ν following $\alpha_{\nu'}$ in Σ and following α_λ. In α_ν there exists a set $A(\nu) \ni x$, contained in $A(\nu')$ and, hence, in all the sets of each of the systems (i), for $i = 1, 2, \dots, s^{\alpha_\lambda}$. Let $A_i^{\alpha_\lambda} = A(\lambda) \in \alpha_\lambda$ be the unique element of the covering α_λ containing $A(\nu)$. Then $A(\lambda)$ is contained in all the elements of the system (i) and in $A_i^{\alpha_\lambda}$—contrary to the supposition.

Thus, by induction, in every α_λ we can choose such an element $A(\lambda) \ni x$ that for any finite number of the $A(\lambda_1), \dots, A(\lambda_r)$ chosen by us

there is an α_ν, inscribed in all $\alpha_{\lambda_1}, \dots, \alpha_{\lambda_r}$, and in it an element $A(\nu) \ni x$, contained in $A(\lambda_1), \dots, A(\lambda_r)$. Hence it follows: if $\alpha_{\lambda'} > \alpha_\lambda$ in Σ, then necessarily $A(\lambda') \subseteq A(\lambda)$. Indeed, for some α_ν following in Σ after $\alpha_{\lambda'}$ (and after α_λ), some $A(\nu) \in \alpha_\nu$ is contained in $A(\lambda')$ and in $A(\lambda)$, i.e. $A(\lambda')$, respectively $A(\lambda)$, is the unique element of the cover $\alpha_{\lambda'}$, respectively α_λ, containing the set $A(\nu)$, whence the assertion $A(\lambda) \supseteq A(\lambda')$ follows at once. For the given point x we have selected one element $A_{i_\alpha}^\alpha$ containing it in each cover α in such a way that, when $\alpha' > \alpha$ in Σ, necessarily $A^{\alpha'}_{i_{\alpha'}} \subseteq A^{\alpha}_{i_\alpha}$, and for $\alpha = \alpha_0$ the selected element is precisely $A^{\alpha_0}_{i^0}$. But then the point $\xi = \{i_\alpha\} \in \overline{D}$ is a distinguished point, with $\xi \in U^{\alpha_0}_{i^0}$ and $f\xi = x$. Equality (2) is proved.

4. The proof of the closedness of the mapping is preceded by

Lemma 2. *Let Φ be a bicompactum lying in D; denote by $U_\alpha\Phi$ the star of this bicompactum in the cover δ_α. Whatever neighborhood $O\Phi$ of the bicompactum Φ in D may be, there exists an α such that $U_\alpha\Phi \subseteq O\Phi$.*

Proof. Let $\overline{O\Phi}$ be such a neighborhood of the bicompactum Φ in $\overline{D}$ that $D \cap \overline{O\Phi} = O\Phi$. We shall prove that there exists an α such that the star $\overline{U_\alpha\Phi}$ of the bicompactum Φ in $\overline{\delta_\alpha}$ is contained in $\overline{O\Phi}$: since $D \cap \overline{U_\alpha\Phi} = U_\alpha\Phi$ and $D \cap \overline{O\Phi} = O\Phi$, Lemma 2 will thereby be proved.

Suppose that for every α the bicompactum $B_\alpha = \overline{U_\alpha\Phi} \setminus \overline{O\Phi} \neq \Lambda$. Then the system $\{B_\alpha\}$ is centered and, consequently, $\bigcap_\alpha B_\alpha \neq \Lambda$; take a point $\xi \in \bigcap_\alpha B_\alpha$; for every α it is contained in some $\overline{U^\alpha_i}$ intersecting Φ; but the $\overline{U^\alpha_i}$ containing the point ξ form a base of this point in $\overline{D}$; therefore $\xi \in [\Phi] = \Phi$. At the same time $\xi \in \overline{D} \setminus \overline{O\Phi}$—we have arrived at a contradiction. Lemma 2 is proved.

5. **The mapping f of the space D onto the space X is closed.** It is required to prove: whatever open set $U \subseteq D$ may be, the set $V = X \setminus f(D \setminus U) = \mathcal{E}(x \in X,\ f^{-1}x \subseteq U)$ of all $x \in X$ for which $f^{-1}x \subseteq U$ is open in X. But if $\Phi = f^{-1}x \subseteq U$, then, by Lemma 2, there exists an α such that $U_\alpha\Phi \subseteq U$. Therefore V is the union of all possible sets of the form $V^*_\alpha = \mathcal{E}(x \in X,\ f^{-1}x \subseteq U^*_\alpha)$, where $U^*_\alpha \subseteq U$ is some set that is the union of certain elements of the cover δ_α, and α is arbitrary. Thus it is enough to prove that any set V^*_α is open, or that its complement

$$X \setminus V^*_\alpha = \mathcal{E}(x \in X,\ f^{-1}x \cap (D \setminus U^*_\alpha) \neq \Lambda) = f(D \setminus U^*_\alpha)$$

is closed. But $D \setminus U^*_\alpha$ is the union of those sets $U^\alpha_i \in \delta_\alpha$ which did not enter as summands into the union U^*_α, so that the set $f(D \setminus U^*_\alpha)$, as the union of a finite number of closed sets $fU^\alpha_i = A^\alpha_i$, is closed in X. The assertion is proved.

6. **The mapping $f : D \to X$ is irreducible.** Otherwise there would exist such a U^α_i that $f(D \setminus U^\alpha_i) = X$. But A^α_i is an element of the canonical cover α, hence there exists a point $x_0 \in A^\alpha_i \setminus \supset \bigcup_{j \neq i} A^\alpha_j$. If $x_0 = f\xi_0,\ \xi_0 = \{i^0_\alpha\}$, then necessarily $i^0_\alpha = i$, i.e. $\xi_0 \in U^\alpha_i$, so that $f(D \setminus U^\alpha_i) \neq X$. Our theorem is completely proved.

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CITED LITERATURE

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2. P. S. Aleksandrov, *Combinatorial Topology*, Moscow–Leningrad, 1947.

Note: Figure translations are in progress. See original paper for figures.